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10 Evans Hall

Fall 2005, Math 55
Final Examination

16 December, 2005
5:00-8:00 PM

1. (18 points, 6 points each.) Short answer questions. A correct answer will get full credit whether or not work is shown. An incorrect answer may get partial credit if work is given that follows a basically correct method.

(a) How many strings of decimal digits (i.e., digits $0, 1, \ldots, 9$) of length 6 are there in which either the first two digits are both 1, or the middle two digits are both 2, or the last two digits are both 3. (As usual, "or" is inclusive.)

(b) Suppose a penny, a nickel and a dime are tossed once, and you are given the coins (if any) that come up heads. Assuming all tosses are fair, and regarding the number of cents you obtain as a random variable X, compute the variance $V(X)$.

(c) What is the value of n when the following program halts?

procedure *almostdouble*
$n := 2$
while $n < 100$
$\quad n := 2n-1$

2. (18 points, 6 points each.) Complete the following definitions. Your definitions do not have to have the same wording as those in the text, but for full credit they should be clear, and be equivalent in meaning to those.

(a) If a is an integer and m is a positive integer, then a **mod** m denotes . . .

(b) If S is a countable set, then a *probability distribution* on S means . . .

(c) A sequence $(a_0, a_1, \ldots)$ of real numbers is said to satisfy a *homogeneous linear recurrence relation of degree* d *with constant coefficients* if there exist real numbers . . . [1] such that for all . . . [2] the equation . . . [3] holds.

3. (21 points, 7 points each.) For each of the items listed below, either *give an example* with the properties stated, or give a brief reason why *no such example exists.*

If you give an example, you do *not* have to prove that it has the property stated; however, your examples should be specific; i.e., even if there are many objects of a given sort, you should name a particular one. If you give a reason why no example exists, don't worry about giving reasons for your reasons; a simple statement will suffice.

(a) A probability distribution on a sample space S, and two events $E,\ F \subseteq S$, such that $p(E|F) > p(E)$.

(b) An expression for 1 as a linear combination of 74 and 999 with integer coefficients.

(c) An integer n such that $7n \equiv 1 \pmod{80}$.

4. (10 points.) Show that there exist distinct prime numbers p_1 and p_2 such that $p_1 \equiv p_2$ (mod 1,000,000).

5. (11 points.) Let f_n denote the nth Fibonacci number. (So $f_0 = 0$, $f_1 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for $n > 1$.)

Show that for every positive integer n,

$$f_{n-1}\, f_{n+1} \; - \; f_n^2 \; = \; (-1)^n.$$

(Suggestion: Express this equation in terms of f_{n-1} and f_n only, and prove it by induction.)

6. (11 points.) Suppose that A and B are tasks, each of which produces a nonnegative integer output. (An example of such a task is that of choosing stamps of various denominations to put on an envelope, where the "output" is the total value of those stamps.) Suppose that for every nonnegative integer n, there are only a finite number, a_n, of ways to perform task A so as to give output n, and likewise only a finite number, b_n, of ways to perform task B so as to give output n. For each nonnegative integer n let c_n be the number of ways one can perform task A followed by task B, so that the *sum* of their outputs is n. Prove the equality of generating functions

$$\Sigma\, c_n x^n \; = \; (\Sigma\, a_n x^n)(\Sigma\, b_n x^n).$$

(This was given in class as a precise formulation of a principle that Rosen uses without explicitly stating it. You should not find it hard to prove, even if you were not there that day. Remember that two power series are equal if and only if for each k they have the same coefficient of x^k.)

7. (11 points.) Let $G = (V, E)$ and $G' = (V', E')$ be simple graphs, i.e., graphs with undirected edges, no more than one edge between any two vertices, and no loops. Let us define $c(G) = (V, \overline{E})$ and $c(G') = (V', \overline{E'})$, where $\overline{E}$ denotes the *complement* of E in the set of all unordered pairs of distinct elements of V, and $\overline{E'}$ the complement of E' in the set of all unordered pairs of distinct elements of V'.

Show that if G is isomorphic to G', then $c(G)$ is isomorphic to $c(G')$.