Midterm III/math110/fall 2003  
Total Score: 100 pts  
Time: 2:10-3:30pm Nov 6

#1. (15%) Please determine if the following statements are correct or not. Please give a brief explanation for each of your answers.
(a). If a linear transformation $T \in \mathcal{L}(V)$ has an eigenvalue $\lambda = 0$, then $T$ cannot be invertible. (3%) 
(b). Let $V = P_n(\mathbb{R})$ and let $T \in \mathcal{L}(V)$ be $T = d/dx$. Then $V$ itself is a $T$-cyclic subspace generated by some $f \in V$. (4%)  
(c). Let $V$ be a complex vector space over $\mathbb{C}$ and let $\langle , \rangle$ be an inner product on $V$. Then $V^\perp$ can never be $V$ itself. (4%) 
(d). Let $T \in \mathcal{L}(V)$ and let $W \subset V$ be a $T$-invariant subspace of $V$. If the characteristic polynomial of $T_W$ splits, then the characteristic polynomial of $T$ also splits. (4%) 

#2. (15%) Let $T: \mathbb{R}^4 \mapsto \mathbb{R}^4$ be a linear transformation defined by $(a, b, c, d) \mapsto T(a, b, c, d) = (a + c, a + c, 2a + 2c, -a - c)$.
(a). Please determine the characteristic polynomial of $T$. (hint: you may find the eigenvalues before you find the characteristic polynomial. What is the rank of $T$? What is its range?) (8%)  
(b). Determine if $T$ is diagonalizable or not. If it is diagonalizable, please determine a basis of eigenvectors. If not, explain why not. (7%) 

#3. (15%) (a). Let $V = P_n(\mathbb{R})$ and let $D \in \mathcal{L}(V)$ be $D = d/dx$. Give a uniform proof that for all $1 \leq i \leq n$ the linear operators $D^i$ are not diagonalizable. (5%)  
(b). Let $V$ and $D$ still be as in (a). Let $T = I_V + D + D^2 + \cdots + D^n$. Please determine the characteristic polynomial of $T$ and show that it splits. (5%) 
(c). Continue the question (b). Find all the eigenspaces of $T$. Determine if $T$ is diagonalizable. (some knowledge in O.D.E. may help) (5%) 

#4. (20%) (a). Let $\lambda_0$ be a root of the characteristic equation of $T \in \mathcal{L}(V)$, $\dim V < \infty$. Show that the eigenspace associated to $\lambda_0$, $E_{\lambda_0} \neq \{0\}$. (7%) 
(b). Let $T \in \mathcal{L}(V)$, $\dim V = n$. Suppose that $T$ has $n$ distinct eigenvalues. Prove that the eigenvectors associated with these $n$ distinct eigenvalues are linearly independent. (13%) 

#5. (20%) Let $V = M_{n \times n}(\mathbb{R})$ be a vector space over $\mathbb{R}$ with the inner product $\langle A, B \rangle = tr(AB^t)$. Define $S_n = \{m|m \in V, m = m^t\}$, $A_n = \{m|m \in V, m^t = -m\}$, $D_n$=the subspace of the diagonal $n \times n$ matrices in $V$, $T_n$= the subspace of the upper triangular matrices in $V$. 
(a). Please list all the pairs of subspaces (from $S_n, A_n, D_n, T_n$) such that the direct sums exist. (6%)  
(b). Let $T: V \mapsto V$ be $T(m) = m^t + m$. Determine the eigenvalues of $T$ and all its eigenspaces. (8%) 
(c). When $n = 3$, find an orthonormal basis of $S_n$. (6%) 

#6. (15%) Please state and prove the Cayley-Hamilton theorem for linear transformations $T \in \mathcal{L}(V)$ on a finite dimensional vector space $V$. (If you use some propositions or theorems proved in the lectures, please state them clearly.) 

Extra Credit: (10%) Under the same setup as in #4, prove that $S_n^\perp = A_n$. 