Mathematics 1A, Fall Semester 2003
Instructor: Garth Dales
October 30, 2003

Midterm Examination 2

Your Name: ____________________________________________________________

Your SID: ______________________________________________________________

Your GSI's Name and Section Time: ________________________________________

Directions:
• Do not open your exam until you are instructed to do so.
• You may not use any external aids during the exam: NO books, NO lecture notes, NO formula sheets, NO cell phones.
• Answers without explanation will not receive credit. You must show and justify your work. If necessary, use the backs of the pages or the extra pages attached to your exam, and indicate you have done so.
• When time is called, you must stop working and close your exam.
• All questions are worth 12 points each (for a total of 60 points).

Good Luck!

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1. (a) \( (4 \text{ points}) \) Let \( P_n \) be a statement that may or may not hold for \( n = 1, 2, 3, \ldots \). State the principle of mathematical induction, which explains what we must show to prove \( P_n \) for each \( n = 1, 2, 3, \ldots \).

(b) \( (8 \text{ points}) \) Let \( f(x) = x^{1/2} \) for \( x > 0 \). Prove that

\[
f^{(n)}(x) = (-1)^{n+1} \frac{(2n - 2)!}{2^{n-1}(n - 1)!} \cdot \frac{1}{x^{(2n-1)/2}}
\]

for \( x > 0 \) and \( n = 1, 2, 3, \ldots \).
2. Calculate the derivatives of each of the following functions. (Show your work, but there is no need to give the reason for each step.)

(a) *(4 points)* \( y = \log (x^4 \sin^2 x) \)

(b) *(4 points)* \( y = (\sin x)^x \)

(c) *(2 points)* \( y = \log (\log x) \)

(d) *(2 points)* \( y = \sinh^2 x \)
3. (a) *(8 points)* The altitude of a triangle is increasing at a rate of 1 cm/min while the area of the triangle is increasing at a rate of 2 cm$^2$/min. At what rate is the base of the triangle changing when the altitude is 10 cm and the area is 100 cm$^2$?

(b) *(4 points)* Use l'Hôpital's rule to calculate

\[
\lim_{x \to 0} \frac{\sin x - e^x + 1}{x^2}
\]
4. (a) \((4 \text{ points})\) Find the (absolute) maximum and minimum values of the function

\[ f(x) = x^3 - 6x^2 + 9x + 2 \]

on the closed interval \([-1, 4]\).

(b) \((3 \text{ points})\) Give a careful statement of Rolle's theorem (but do not give the proof).
(c) (5 points) Let $f$ and $g$ be continuous functions on the closed interval $[a, b]$ such that $f$ and $g$ are differentiable on the open interval $(a, b)$. Suppose that $g'(x) \neq 0$ for each $x$ with $a < x < b$. Deduce from Rolle's theorem that there exists $c$ with $a < c < b$ such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$
5. *(12 points)* Sketch the graph of the function

\[ f(x) = \frac{x^2}{x^2 + 3} \quad (x \in \mathbb{R}). \]

In particular, find any intercepts, symmetry, any local maxima and minima, any points of inflection, and any asymptotes to the curve. Coordinate axes are provided for you on the next page.