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Fall 2003, Math 110, Sec. 1
Second Midterm

3 Nov., 2003
3:10-4:00

1. (34 points: 16+9+9.) Complete the following definitions. You may use, without defining them, terms or symbols which our text defined before it defined the word or symbol asked for. Your definitions do not have to have exactly the same form as those in the text, but for full credit they should be clear, and be logically equivalent to those.

(a) If $F = R$ or C, and V is a vector space over F, then an *inner product* on V means a function $\langle\ ,\ \rangle$ such that for all $x, y \in$ _____ we have $\langle x, y\rangle \in$ _____, and satisfying the following laws for all $x, y, z \in \mathsf{V}$ and $c \in F$: $\langle x+y, z\rangle =$ _____. $\langle cx, y\rangle =$ _____. Relation between $\langle x, y\rangle$ and $\langle y, x\rangle$: _____. $\langle x, x\rangle$ _____.

(b) If V is an inner product space and T a linear operator on V, then by the *adjoint* of T we mean the linear operator T^* on V (unique if it exists) such that . . .

(c) If V is an inner product space over C, then a linear operator T on V is said to be *unitary* if . . .

2. (27 points; 9 points each.) For each of the items listed below, either *give an example* with the properties stated, or give a brief reason why *no such example exists.*

If you give an example, you do *not* have to prove that it has the property stated; however, your examples should be specific; i.e., even if there are many objects of a given sort, you should name a particular one. If you give a reason why no example exists, don't worry about giving reasons for your reasons; a simple statement will suffice.

(a) A square matrix A over R with no eigenvectors.

(b) A square matrix A over C with no eigenvectors.

(c) A 3×3 matrix A over a field F, such that the matrices I_3, A, A^2, A^3 and A^4 are linearly independent.

3. (39 points; 13 points each.) Computational proofs. Parts (a) and (b) ask you to prove results given in the text; in these cases you may use in your proofs only results appearing earlier in the text than the stated results. In part (c) you may use any results appearing in the parts of the text we have read.

(a) Show that if $\{v_1, \dots, v_k\}$ is an orthogonal basis of an inner product space V, and $x \in \mathsf{V}$, then $x = \Sigma_{j=1}^{k} (\langle x, v_j\rangle / \|v_j\|^2)\, v_j$.

(b) Show that if U and T are operators on a finite-dimensional inner product space V, then $(\mathsf{UT})^* = \mathsf{T}^*\mathsf{U}^*$. (This is stated as part of a theorem in the text, with the proof of that part left to the reader.)

(c) Suppose A and B are similar $n\times n$ matrices over C, where "similar" means that there exists an invertible matrix Q over C such that $B = Q^{-1}AQ$. Show that $\lim_{m\to\infty} A^m = 0$ if and only if $\lim_{m\to\infty} B^m = 0$. If you use some properties of limits of matrices and/or similarity of matrices, indicate what properties you are using.