Problem 1. Start with the basic feasible vector \( x^T = [0, 1, 1] \) and then use the \textit{simplex tableau method} to solve
\[
(P) \quad \begin{cases} 
\text{minimize } x_1 + x_2 + 3x_3, \text{ subject to the constraints} \\
\quad x_1 + x_3 = 1, \quad x_2 + x_3 = 2, \quad x_1 \geq 0, x_2 \geq 0, x_3 \geq 0. 
\end{cases}
\]
In particular, demonstrate how to set up the initial tableau and then how to modify it by pivoting, to get an optimal solution.

Problem 2. The vector \( x^T = [\frac{4}{3}, \frac{10}{3}, 0, 0] \) is an optimal solution of:
\[
(P) \quad \begin{cases} 
\text{minimize } 3x_1 + 2x_2, \text{ subject to the constraints} \\
\quad 2x_1 + x_2 - x_3 = 6, \quad x_1 + 2x_2 - x_4 = 8, \\
\quad x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0. 
\end{cases}
\]
Use the equilibrium equations to find an optimal solution \( y^T = [y_1, y_2] \) of the dual problem \((\bar{D})\).

Problem 3. Prove that if there exists a solution \( x \) of
\[
(*) \quad Ax = b, \quad x \geq 0,
\]
then there exists a basic solution.
(Hint: Take a solution \( x \) of \((*)\) with the fewest number of nonzero components. If the columns \( \{a^\alpha, a^\beta, \ldots, a^\sigma\} \) of \( A \) corresponding to the nonzero entries in \( x \) are dependent, we can find \( \{\theta_\alpha, \theta_\beta, \ldots, \theta_\sigma\}, \) not all zero, such that \( \theta_\alpha a^\alpha + \theta_\beta a^\beta + \cdots + \theta_\sigma a^\sigma = 0. \) Finish the proof from here.)

Problem 4. Let \( A \) be an \( m \times n \) matrix.
(a) State the \textit{Farkas alternative} (ii) to the assertion
\[
(i) \quad Ax = b, \quad x \geq 0 \quad \text{has a solution } x.
\]
(b) Use the Separating Hyperplane Theorem to prove that either (i) or (ii) holds, but not both.
(Hint: You may assume that the finite cone \( C = \{Ax \mid x \geq 0\} \) is closed and convex.)

Problem 5. Suppose \( A \) is a symmetric \( n \times n \) matrix; that is, \( A^T = A. \) Consider the linear programming problem:
\[
(P) \quad \text{minimize } b \cdot x, \quad \text{subject to } Ax = b, \quad x \geq 0.
\]
Show that any feasible \( x \) is in fact optimal.