George M. Bergman
75 Evans Hall

Fall 2003, Math 110, Sec. 1
First Midterm

29 Sept., 2003
3:10-4:00

1. (24 points, 8 points each.) Find the following.

(a) $\dim(\mathrm{span}(\{ \begin{pmatrix}1\\1\\1\\-3\end{pmatrix}, \begin{pmatrix}1\\1\\-3\\1\end{pmatrix}, \begin{pmatrix}1\\-3\\1\\1\end{pmatrix}, \begin{pmatrix}-3\\1\\1\\1\end{pmatrix} \}))$, where the given columns lie in R^4.

(b) $\dim(\mathsf{P}_{10}(R)^*)$; i.e., the dimension of the dual space of the space of polynomials of degree ≤ 10 over the real numbers.

(c) Two linearly independent eigenvectors of the matrix $\begin{pmatrix}-1 & 0\\ 1 & 2\end{pmatrix} \in \mathsf{M}_{2\times 2}(R)$.

2. (24 points, 8 points each.) Complete the following definitions. You may use, without defining them, terms or symbols which our text defined before it defined the word or symbol asked for. Your definitions do not have to use exactly the same words as those in the text, but for full credit they should be clear, and be logically equivalent to those.

(a) If A is an $m\times n$ matrix over a field F, then the L_A (the "left-multiplication transformation by A") denotes . . .

(b) If V is a vector space over a field F, then a *basis* for V means . . .

(c) If V and W are vector spaces over a field F, then an *isomorphism* from V onto W means . . .

3. (24 points; 8 points each.) For each of the items listed below, either *give an example,* or give a brief reason why *no example exists.*

If you give an example, you do *not* have to prove that it has the property stated; however, your examples should be specific; i.e., even if there are many objects of a given sort, name a particular one. If you give a reason why no example exists, don't worry about giving reasons for your reasons; a simple statement will suffice.

(a) An infinite-dimensional vector space over the real numbers.

(b) A one-to-one linear transformation $\mathsf{T}: \mathsf{R}^5 \to \mathsf{R}^4$.

(c) Two distinct similar matrices in $\mathsf{M}_{2\times 2}(C)$.

4. (14 points.) Let V, W, Z be vector spaces over a field F, and let $\mathsf{T}: \mathsf{V} \to \mathsf{W}$ and $\mathsf{U}: \mathsf{W} \to \mathsf{Z}$ be linear transformations. Show that $\mathsf{UT} = \mathsf{T}_0$ if and only if $\mathsf{R(T)} \subseteq \mathsf{N(U)}$. (Here T_0 denotes the zero element of $\mathcal{L}(\mathsf{V}, \mathsf{Z})$.)

5. (14 points.) Show that if V is an n-dimensional vector space (where n is a positive integer), W another vector space, and W' an n-dimensional subspace of W, then there exists a linear transformation $\mathsf{T}: \mathsf{V} \to \mathsf{W}$ whose range is exactly W'.