1. (15%) Determine if the following statements are true or false and give a brief reasoning for each statement.

(a). Every vector space $V$ over $\mathbb{R}$ is finite dimensional. (3%)  
(b). The union of any two vector spaces over $\mathbb{R}$ is always a vector space over $\mathbb{R}$. (4%)  
(c). The set $V = \{(x, y) | x, y \in \mathbb{R}\}$ with the addition $(x_1, y_1) + (x_2, y_2) = (y_1 + x_2, x_1 + y_2)$ and scalar multiplication $c \cdot (x, y) = (cx, cy)$, $c \in \mathbb{R}$ forms a vector space over $\mathbb{R}$. (4%)  
(d). Any basis of an $n$ dimensional vector space over $\mathbb{R}$ can not contain more than $n$ elements. (4%)  

2.(20%) Let $S = \{v_1, v_2, v_3, v_4, v_5\}$, with $v_1 = 1 + x + x^2$, $v_2 = 1 - 2x$, $v_3 = x + 3x^2$, $v_4 = 1 + x^2$, $v_5 = -2 - x - x^2$, be a set of five vectors in $P_2(\mathbb{R})$.  
(a). Please show that Span$(S) = P_2(\mathbb{R})$. (10%)  
(b). Please find a subset of $S$ which forms a basis of $P_2(\mathbb{R})$. Please explain why your choice is a basis. (10%)  

3. (25%) Let $V = M_{n \times n}(\mathbb{R})$ be the space of $n$ by $n$ matrices. Consider the subset $W_n$ of skew-symmetric matrices, i.e. the set of $n \times n$ matrices $m$ in $V$ such that $m^t = -m$. ($m^t$ means the transpose of $m$)  
(a). Show that $W$ is a subspace. (10%)  
(b). For $W_3 \subset M_{3 \times 3}(\mathbb{R})$, find a basis of $W_3$ and justify that it is a basis of $W_3$. (10%)  
(c). Determine the dimension of $W_3$. (5%)  

4.(20%) (a). Consider a few functions  
$T_1 : \mathbb{R}^2 \rightarrow \mathbb{R}^3$, $T_1(x, y) = (x + y, y, x - y)$.  
$T_2 : \mathbb{R}^2 \rightarrow \mathbb{R}^3$, $T_2(x, y) = (2x + 1, 4y - 3, x)$.  
$T_3 : P_3(\mathbb{R}) \rightarrow P_2(\mathbb{R})$, $T_3(f) = f^2$, $f \in P_3(\mathbb{R})$.  
$T_4 : M_{2 \times 2}(\mathbb{R}) \rightarrow \mathbb{R}$, $T_4(m) = tr(m)$, for $m \in M_{2 \times 2}(\mathbb{R})$.  
$T_5 : M_{3 \times 3}(\mathbb{R}) \rightarrow M_{3 \times 2}(\mathbb{R})$ defined by $T_5(m) = m - m^t$ for $m \in M_{3 \times 3}(\mathbb{R})$.  
List all those which are not linear transformations. For each of them, explain why it is not a linear transformation. (12%)  
(b). Let $\{v_1, v_2, \ldots, v_n\}$ be a linear dependent subset of $V$ and let $T : V \rightarrow W$ be a linear transformation. Show that $\{T(v_1), T(v_2), \ldots, T(v_n)\}$ is a linear dependent subset of $W$. (8%)  

5.(20%) Let $S = \{v_1, v_2, \ldots, v_n\}$ be a basis of a vector space $V$.  
(a). Prove that (without applying theorems) for any $v \in V$, but $v \notin S$, the union $S \cup \{v\}$ is linearly dependent. (10%)  
(b). Prove that (without applying theorems) for any $v \in S$, the deletion of $v$ from $S$, $S - \{v\}$, does not generate $V$. (10%)  

Extra Credit:  
(10%) Let $W \subset V$ be a subspace of the finite dimensional vector space $V$. Prove that $dim(W) \leq dim(V)$.  

The 8 Axioms of Vector Spaces over $\mathbb{R}$

Let $V$ be a set with an addition $+$ and a scalar multiplication $\cdot$. $V$ is said to be a vector space over $\mathbb{R}$ if

(VS1) For all $x, y \in V$, $x + y = y + x \in V$.

(VS2) For all $x, y, z \in V$, $(x + y) + z = x + (y + z)$.

(VS3) There exists an element in $V$ denoted by $\mathbf{0}$ such that $x + \mathbf{0} = x$ for each $x \in V$.

(VS4) For each $x \in V$, $\exists y \in V$ such that $x + y = \mathbf{0}$.

(VS5) For all $x \in V$, $1 \cdot x = x$.

(VS6) For each pair of $a, b \in \mathbb{R}$ and each element $x \in V$, $(ab) \cdot x = a \cdot (b \cdot x)$.

(VS7) For each $a \in \mathbb{R}$ and each pair of $x, y \in V$, $a \cdot (x + y) = a \cdot x + a \cdot y$.

(VS8) For each pair of $a, b \in \mathbb{R}$ and each element $x \in V$, $(a + b) \cdot x = a \cdot x + b \cdot x$. 
