MATH H110

PROFESSOR KENNETH A. RIBET

Final Examination

December 16, 2003

5–8 PM

70 Evans

Name:

Please put away all books, calculators, electronic games, cell phones, pagers, .mp3 players, PDAs, and other electronic devices. You may refer to a single 2-sided sheet of notes. Please write your name on each sheet of paper that you turn in; don't trust staples to keep your papers together. Explain your answers in full English sentences as is customary and appropriate. Your paper is your ambassador when it is graded.

Problem	Your score	Total points
1		5 points
2		6 points
3		6 points
4		5 points
5		6 points
6		6 points
7		5 points
8		6 points
Total:		45 points

1. Let A be an $n \times n$ matrix. Suppose that there is a non-zero row vector y such that $yA = y$. Prove that there is a non-zero column vector x such that $Ax = x$. (Here, A, x and y have entries in a field F.)

2. Let A and B be $n \times n$ matrices over a field F. Suppose that $A^2 = A$ and $B^2 = B$. Prove that A and B are similar if and only if they have the same rank.

3. Suppose that $T\colon V \to V$ is a linear transformation on a finite-dimensional real inner product space. Let T^* be the adjoint of T. Show that V is the direct sum of the null space of T and the range of T^*.

4. Let A be a symmetric real matrix whose square has trace 0. Show that $A = 0$.

5. Let $T\colon V \to W$ be a linear transformation between finite-dimensional vector spaces. Let X be a subspace of W. Let $T^{-1}(X)$ be the set of vectors in V that map to X. Show that $T^{-1}(X)$ is a subspace of V and that $\dim T^{-1}(X) \geq \dim V - \dim W + \dim X$.

7. Let T be a nilpotent operator on a finite-dimensional complex vector space. Using the table

i	0	1	2	3	4	5	$\cdots$
nullity(T^i)	0	4	7	9	10	10	$\cdots$,

find the Jordan canonical form for T.

6. Suppose that V is a real finite-dimensional inner product space and that $T\colon V \to V$ is a linear transformation with the property that $\langle T(x), T(y)\rangle = 0$ whenever x and y are elements of V such that $\langle x, y\rangle = 0$. Assume that there is a non-zero $v \in V$ for which $\|T(v)\| = \|v\|$. Show that T is orthogonal.

8. Let F be a finite field; write q for the number of elements of F. Let V be an n-dimensional vector space over F. Compute, in terms of n and q, the number of 1-dimensional subspaces of V and the number of linear transformations $V \to V$ that have rank 1.