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Fall 2003, Math 104, Sec. 2
Final Exam

10 Dec., 2003
12:30-3:30

1. (32 points, 4 points each.) Complete the following definitions. Unless otherwise stated you may use without defining them any terms or symbols which Rudin defines before he defines the concept asked for. You do not have to use exactly the same words as Rudin, but for full credit your statements should be clear, and be logically equivalent to his.

(a) Let X be a set, and let $E_1, E_2, \ldots, E_n, \ldots$ be countably many subsets of X. Then $\bigcap_{n=1}^{\infty} E_n$ denotes the set of those elements $x \in X$ such that . . .

(b) If (p_n) (or in Rudin's notation, $\{p_n\}$) is a sequence of elements of a set X, then a *subsequence* of (p_n) means . . .

(c) If X and Y are metric spaces, then a function $f: X \to Y$ is said to be *continuous* at a point $p \in X$ if . . .

(d) If f is a real-valued function on an interval $[a, b]$ and x is a point of $[a, b]$, then by the derivative $f'(x)$ (if this exists) we mean the real number . . .

(e) If f is a real-valued function on an interval $[a, b]$, α an increasing function on $[a, b]$, and $P = \{x_0, x_1, \ldots, x_n\}$ a partition of $[a, b]$, then the *lower sum* $L(P, f, \alpha)$ is defined to be . . . (If your definition uses symbols for certain numbers defined in terms of f, P and α, indicate what they mean.)

(f) If X is a metric space, then $\mathscr{C}(X)$ denotes the metric space described as follows: The points of $\mathscr{C}(X)$ are . . .
The distance between two such points is given by . . .

(g) A family $\mathscr{A}$ of complex-valued functions on a set E is said to be an *algebra* if . . .

(h) A family $\mathscr{A}$ of functions on a set E is said to *separate points* on E if . . .

2. (32 points, 4 points each.) For each of the items listed below, either *give an example* with the properties stated, or give a brief reason why *no such example exists.*

If you give an example, you do *not* have to prove that it has the property stated; however, your examples should be specific; i.e., even if there are many objects of a given sort, name a particular one. If you give a reason why no example exists, don't worry about giving a detailed proof; the key relevant fact will suffice.

(a) A subset E of the ordered field Q of rational numbers which has an upper bound in Q but does not have a least upper bound in Q.

(b) A bounded sequence in R with no convergent subsequence.

(c) A continuous one-to-one and onto function between metric spaces, $f: X \to Y$, such that the inverse function $f^{-1}: Y \to X$ is not continuous.

(d) A real-valued function f on $[0, 1]$ which is Riemann integrable, but such that the function $|f|$ defined by $|f|(x) = |f(x)|$ is not Riemann integrable.

(e) A family of complex numbers $s_{m,n}$ $(m, n > 0)$ such that $\lim_{m\to\infty} (\lim_{n\to\infty} s_{m,n})$ and $\lim_{n\to\infty} (\lim_{m\to\infty} s_{m,n})$ are both defined, but are unequal.

(f) A set E and a sequence of real-valued functions f_n on E which are pointwise convergent but not uniformly convergent.

(g) A real-valued function f on $[0,1]$ which is not a uniform limit of polynomials.

(h) A continuous real-valued function f on $(0,1)$ which is not a uniform limit of polynomials.

3. (4 points.) For $\mathbf{x}, \mathbf{y} \in R^k$, prove that $|\mathbf{x}+\mathbf{y}| \geq |\mathbf{x}| - |\mathbf{y}|$. You may assume inequalities proved in Rudin.

4. (6 points.) Suppose (p_n) is a Cauchy sequence in a metric space X, and suppose some subsequence (p_{n_k}) converges to a point $p \in X$. Prove that (p_n) also converges to p.

5. (8 points.) Most of Rudin's proof of Weierstrass's Theorem is devoted to showing that if f is a continuous complex-valued function on $[0,1]$, then there exists a sequence of polynomials P_n such that $P_n(x) \to f(x)$ uniformly on $[0,1]$. Show that once we know this fact, we can deduce from it the statement of the theorem, namely that if f is a continuous complex-valued function on *any* interval $[a,b]$ (for $a < b$ real numbers) then there exists a sequence of polynomials P_n such that $P_n(x) \to f(x)$ uniformly on $[a,b]$.

Your proof will involve some way of obtaining from functions on $[a,b]$ functions on $[0,1]$ and vice versa. Be precise in showing that a function of one sort does indeed give a function of the other sort, and that the uniform convergence assumed implies the uniform convergence desired.

(As discussed in class, such an argument is what Rudin is assuming the reader can supply to when he says "We may assume, without loss of generality, that $[a,b] = [0,1]$". Very little but definitions are needed for this argument; but, just to set down the rules: You may use any facts proved in Rudin before Weierstrass's Theorem, but not that theorem or anything proved later. Note also that, for brevity, I am *not* also asking you to prove, as Rudin does, that we can also assume $f(0) = f(1) = 0$.)

6. (18 points, 3 points each.) Below, a theorem is proved. After certain steps of the proof I have inserted parenthetical questions such as "($\boxed{0}$ Why?)". Answer each of these questions at the bottom of the page, after the corresponding number. Your answers can be results proved in Rudin (you don't have to specify their statement-numbers!), observations about the given situation, or calculations. You should seldom need as much space as is given for the answers; one key fact or calculation is what is wanted in each case. Note also that if you can't justify some step, you may still assume it in justifying later steps.

Theorem. *Let E be a subset of a metric space X, and $f: E \to R$ a uniformly continuous function. Then there exists a uniformly continuous function $F: \bar{E} \to R$ which extends f. (Here $\bar{E}$ denotes the closure of E in X, and the statement that F extends f means that $F(p) = f(p)$ for all $p \in E$.)*

Proof. For each $p \in \bar{E}$, let us choose a sequence (p_n) in E such that $\lim_{n\to\infty} p_n = p$. (If p is a limit point of E this can be done by Theorem 3.2(d); if p is an isolated point of E, we can take (p_n) to be the constant sequence with all $p_n = p$.) We claim that $(f(p_n))$ is a Cauchy sequence in R. Indeed, given $\varepsilon > 0$, let us take a δ such that for all points $x, y \in E$ with $d(x,y) < \delta$ we have $|f(x) - f(y)| < \varepsilon$. ($\boxed{1}$ What condition assumed above implies that such δ exists?) Choose an N such that for $n \geq N$ we have $d(p_n, p) < \delta/2$. ($\boxed{2}$ What condition assumed above implies that such N exists?) Then for $m, n \geq N$ we have $|f(p_m) - f(p_n)| < \varepsilon$. ($\boxed{3}$ Why?)

Hence $(f(p_n))$ converges to some real number; let us call this $F(p)$. Doing this for each $p \in \bar{E}$, we get a function $F: \bar{E} \to R$. We see that if $p \in E$, then $F(p) = f(p)$. ($\boxed{4}$ Why?) That is, F extends f, as claimed.

To show that F is uniformly continuous, again let $\varepsilon > 0$. Take any positive $\varepsilon' < \varepsilon$, and let us now choose $\delta > 0$ (not, in general, the δ of the preceding paragraph) such that for all $p, q \in E$ with $d(p,q) < \delta$ we have $|f(p) - f(q)| < \varepsilon'$. We shall show that for $p, q \in \bar{E}$ with $d(p,q) < \delta$ we have $|F(p) - F(q)| < \varepsilon$.

Indeed, given $p, q \in \bar{E}$ with $d(p,q) < \delta$, let (p_n) and (q_n) be the sequences used in defining $F(p)$ and $F(q)$ as above. Let us choose a positive integer m satisfying both $d(p_m, p) < (\delta - d(p,q))/2$ and $|f(p_m) - F(p)| < (\varepsilon - \varepsilon')/2$. (By choice of (p_m) the first inequality will hold for all $m \geq$ some M_1 and by definition of $F(p)$ the second will hold for all $m \geq$ some M_2, hence there will be some m for which both hold. ($\boxed{5}$ For instance?)) Similarly, let us choose n satisfying both $d(q_n, q) < (\delta - d(p,q))/2$ and $|f(q_n) - F(q)| < (\varepsilon - \varepsilon')/2$.

Putting together our inequalities involving distances between points of $\bar{E}$, we see that $d(p_m, q_n) < \delta$. ($\boxed{6}$ Show computation.) Hence by choice of δ, $|f(p_m) - f(q_n)| < \varepsilon'$, hence $|F(p) - F(q)| \leq |F(p) - f(p_m)| + |f(p_m) - f(q_n)| + |f(q_n) - F(q)| \leq (\varepsilon - \varepsilon')/2 + \varepsilon' + (\varepsilon - \varepsilon')/2 = \varepsilon$. Since we have proved this for any $p, q \in \bar{E}$ with $d(p,q) < \delta$, we have proved F uniformly continuous, as required.