George M. Bergman
6 Evans Hall

Fall 2003, Math 110, Sec. 1

Final Exam

12 Dec., 2003
12:30-3:30

1. (15 points, 5 points each.) Find the following.

(a) A real number c such that the vectors $\begin{pmatrix}1\\2\\4\end{pmatrix}, \begin{pmatrix}2\\3\\5\end{pmatrix}, \begin{pmatrix}2\\7\\c\end{pmatrix}$ do *not* form a basis of $\mathbb{R}^3$.

(b) $\lim_{n\to\infty} A^n$, where $A = \begin{pmatrix}1 & -3\\2 & -5\end{pmatrix}\begin{pmatrix}1 & 0\\0 & -\frac{1}{2}\end{pmatrix}\begin{pmatrix}1 & -3\\2 & -5\end{pmatrix}^{-1}$.

(c) A finite list of 3×3 matrices over R such that every 3×3 matrix over R with characteristic polynomial $-(t-3)(t+2)^2$ is similar to one and only one matrix in your list.

2. (24 points, 4 points each.) Complete the following definitions. You may use, without defining them, terms or symbols which our text defines before it defines the word or symbol asked for. Your definitions do not have to have exactly the same form as those in the text, but for full credit they should be clear, and be logically equivalent to those.

(a) If V is a vector space over a field F, then a *subspace* of V means . . .

(b) Two $n\times n$ matrices A and B over the same field F are said to be *similar* if . . .

(c) If V is a vector space and $W_1, \ldots, W_k$ are subspaces of V, then we write $V = W_1 \oplus \ldots \oplus W_k$ (and say that V is the *direct sum* of the subspaces $W_1, \ldots, W_k$) if . . .

(d) If A is a square matrix over a field F, and $\lambda \in F$ is an eigenvalue of A, then the *multiplicity* of λ as an eigenvalue of A means . . .

(e) If V is an inner product space, then a projection map $T: V \to V$ is called an *orthogonal projection* if . . .

(f) A linear operator T on a finite-dimensional inner product space V is said to be *normal* if . . .

3. (20 points, 5 points each.) For each of the items listed below, either *give an example* with the properties stated, or give a brief reason why *no such example exists.*

If you give an example, you do *not* have to prove that it has the property stated; however, your examples should be specific; i.e., even if there are many objects of a given sort, you should name a particular one. If you give a reason why no example exists, don't worry about giving reasons for your reasons; a simple statement will suffice.

(a) An isomorphism between the real vector spaces $M_{3\times 3}(R)$ and $\mathbb{R}^3$.

(b) An isomorphism between the real vector spaces $M_{3\times 3}(R)$ and $\mathbb{R}^9$.

(c) A unitary operator U on a finite-dimensional complex inner product space having $1+i$ as an eigenvalue.

(d) A rigid motion $f\colon \mathsf{R}^2 \to \mathsf{R}^2$ which is not a linear operator.

4. (11 points.) Let V be a vector space, let T be a linear operator on V, and let W be a subspace of V. Recall that W is said to be T-invariant if for every $x \in \mathsf{W}$ we have $\mathsf{T}(x) \in \mathsf{W}$. Show that for any $\lambda \in F$, the subspace W is T-invariant if and only if it is $(\mathsf{T}-\lambda\, \mathsf{I_V})$-invariant.

5. (12 points) Let $x_1, \ldots, x_n$ be an orthogonal set of distinct nonzero elements in an inner product space V. Show that $x_1, \ldots, x_n$ are linearly independent.

6. (18 points, 3 points each) Suppose V is a 5-dimensional vector space over a field F, and suppose T is a linear operator on V, which has characteristic polynomial $-(t-\lambda)^5$ for some $\lambda \in F$, and such that $\operatorname{rank}(\mathsf{T}-\lambda\mathsf{I}) = 2$ while $(\mathsf{T}-\lambda\mathsf{I})^2 = \mathsf{T}_0$.

Below, we will discuss how to construct a Jordan basis for T. After certain steps of the discussion I have inserted parenthetical questions such as "($\boxed{0}$ Why?)". Answer each of these questions at the bottom of the page, after the corresponding number. Your answers can be results proved in the text (you don't have to specify the theorem-number!), observations about the given situation, calculations, etc.. You should seldom need as much space as is given for the answers; one key fact or calculation is generally what is wanted. If you can't justify some step, you may still assume it in justifying later steps.

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Given V and T as described in the first paragraph above, let $\{v_1, v_2\}$ be a basis for $\mathsf{R}(\mathsf{T}-\lambda\mathsf{I})$. ($\boxed{1}$ What assumption implies that it has a 2-element basis?) Now $\mathsf{N}(\mathsf{T}-\lambda\mathsf{I})$ is 3-dimensional. ($\boxed{2}$ Why?) Moreover, $\mathsf{R}(\mathsf{T}-\lambda\mathsf{I}) \subseteq \mathsf{N}(\mathsf{T}-\lambda\mathsf{I})$. ($\boxed{3}$ What assumption implies this?) Hence we can extend the basis $\{v_1, v_2\}$ of $\mathsf{R}(\mathsf{T}-\lambda\mathsf{I})$ to a basis $\{v_1, v_2, v_3\}$ of $\mathsf{N}(\mathsf{T}-\lambda\mathsf{I})$.

Let us also choose $v_4, v_5 \in \mathsf{V}$ such that $v_1 = (\mathsf{T}-\lambda\mathsf{I})(v_4)$ and $v_2 = (\mathsf{T}-\lambda\mathsf{I})(v_5)$. ($\boxed{4}$ Why do there exist such elements?) By a result in the book, the vectors v_1, v_2, v_3, v_4, v_5 are distinct and linearly independent. ($\boxed{5}$ What does that result in the book say? You don't have to explain why it applies to this case.) Hence they form a basis for the 5-dimensional space V. Being a union of disjoint cycles, it is a Jordan basis, and it clearly corresponds to the dot diagram $\begin{smallmatrix}\bullet & \bullet & \bullet\\ \bullet & \bullet & \end{smallmatrix}$. ($\boxed{6}$ Label the dot diagram at the bottom of the page to show which basis vector is represented by each dot.)