Math 110, Section 7
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FINAL EXAMINATION

Open book, open notes. In your solutions, you may use any result from the lectures, from the material we covered in the textbook, or from the assigned homework exercises.

The points for each question are in parenthesis.

1. (20) (a) Express the general vector $v = (a, b)$ in $\mathbb{R}^2$ as a linear combination of the vectors $v_1 = (1, 0)$ and $v_2 = (1, 1)$.

   (b) By means of (a) (or by some other method) find an operator P on $\mathbb{R}^2$ such that $P^2 = P$ and $P^* \neq P$.

2. (20) Give examples of

   (a) a normal operator on $\mathbb{R}^3$ that is not self-adjoint;

   (b) a nilpotent operator on $\mathbb{R}^5$ with index of nilpotency 2;

   (c) two operators S and T on $\mathbb{R}^2$ such that $ST = 0$ but $TS \neq 0$;

   (d) two nilpotent operators S and T on $\mathbb{R}^2$ such that $S + T$ is invertible.

3. (20) Let V be a three-dimensional inner product space. Let e_1, e_2 be an orthonormal pair of vectors in V, and let the vector v in V satisfy $\langle v, e_1\rangle = 3$, $\langle v, e_2\rangle = 4$, and $\|v\| = 13$. Let the operator T in $\mathcal{L}(V)$ satisfy $Te_1 = v$, $Te_2 = v$, $Tv = v$.

   (a) Prove e_1, e_2, v is a basis for V, and apply the Gram–Schmidt procedure to this basis to produce an orthonormal basis e_1, e_2, e_3.

   (b) Find the matrix for T with respect to the basis e_1, e_2, e_3.

4. (10) Let V be a finite-dimensional inner product space, let T be a positive operator in $\mathcal{L}(V)$, and let S be an isometry in $\mathcal{L}(V)$. Prove the operator $S^{-1}TS$ is positive.

5. (20) Let V be a finite-dimensional inner product space, and let P and Q be orthogonal projections in $\mathcal{L}(V)$ such that $PQ = QP$. Let $R = P + Q - PQ$.

   (a) Prove R is an orthogonal projection.

   (b) Prove range $R =$ range $P+$ range Q.

6. (10) Let V be a vector space of finite dimension n. Let the operator T in $\mathcal{L}(V)$ have n distinct eigenvalues $\lambda_1, \ldots, \lambda_n$. Let S be an operator in $\mathcal{L}(V)$ that commutes with T. Prove there is a polynomial p in $\mathcal{P}(\mathbb{F})$ such that $S = p(T)$.

7. (10) Let V be a complex vector space of dimension 6. Let the operator T in $\mathcal{L}(V)$ have minimal polynomial $q_T(z) = (z^2+1)^2$ and characteristic polynomial $p_T(z) = (z^2+1)^3$. Write down all possible Jordan matrices for T, and justify your answer.

8. (20) Let V be a complex vector space of finite dimension n, and let T be an operator in $\mathcal{L}(V)$. Consider the two subspaces

$$\alpha(T) = \{X \in \mathcal{L}(V) : TX = XT\}$$

$$\beta(T) = \{TX - XT : X \in \mathcal{L}(V)\}$$

of $\mathcal{L}(V)$, called, respectively, the commutant and the commutator of T.

(a) Prove $\dim \alpha(T) + \dim \beta(T) = n^2$

(b) Prove $\dim \alpha(T) \geq n$.