1. (15%) Please determine if the following statements are true or not. Give a brief reasoning for each of your answers.
   (a) All the unitary operators on a finite dimensional vector space over $\mathbb{C}$ are normal. (3%) 
   (b) All $n \times n$ matrices in $M_{n \times n} (\mathbb{R})$ have their associated Jordan canonical forms over $\mathbb{R}$. (3%) 
   (c) Let $V$ be a finite dimensional inner product space and $T \in \mathcal{L}(V)$. Then $R(T^*) = N(T)$. (3%) 
   (d) Let $V$ be a finite dimensional inner product space. An operator $T \in \mathcal{L}(V)$ is self-adjoint iff $[T]_{\beta} = [T]_{\beta}^*$ for all the ordered bases $\beta$. (3%) 
   (e) All the orthogonal transformations on a finite dimensional vector space over $\mathbb{R}$ are onto. (3%) 

2. (20%) (a) Prove Schur's theorem, i.e. when the characteristic polynomial of a linear transformation $T \in \mathcal{L}(V)$ (on a finite dimensional inner product vector space $V$ over $F = \mathbb{R}$ or $\mathbb{C}$) splits, then there exists an orthonormal basis $\beta$ such that $[T]_{\beta}$ is upper-triangular. (14%) 
   (b) Prove that a self-adjoint operator $T$ over $\mathbb{C}$ must be diagonalizable by an orthonormal basis. i.e. $\exists$ an orthonormal basis $\beta$ such that $[T]_{\beta}$ is diagonal. (6%) 

3. (25%) (a) Consider the matrix $A = \begin{pmatrix} 0 & 4 & 0 \\ 4 & 0 & 4 \\ 0 & 4 & 0 \end{pmatrix}$. 
   Show that $A$ is normal and diagonalize $A$ by an orthogonal matrix. (10%) 
   (b) For the same $A$, write $I_n = \lambda_1 T_1 + \lambda_2 T_2 + \lambda_3 T_3$ and find $T_1, T_2, T_3$ explicitly. (5%) 
   (c) Let $T : P_3(\mathbb{R}) \rightarrow P_4(\mathbb{R})$ be $T(f) = f'' + f$. Determine the dot diagram and write down the Jordan canonical form of $T$. (10%) 

4. (20%) (a) Let $U$ be an unitary operator upon an inner product space $(V, \langle \cdot, \cdot \rangle)$ over $\mathbb{R}$, i.e. $||U(x)|| = ||x||$ for all $x \in V$. 
   Prove that $\langle U(x), U(y) \rangle = \langle x, y \rangle$ for all $x, y$. (6%) 
   (b) Suppose that $T_1, T_2 \in \mathcal{L}(V)$ are linear operators over an inner product space $(V, \langle \cdot, \cdot \rangle)$ such that the identity $\langle x, T_1(y) \rangle = \langle x, T_2(y) \rangle$ holds for all $x, y \in V$. Show that $T_1 = T_2$. (5%) 
   (c) Let $W$ be a finite dimensional subspace of the inner product space $(V, \langle \cdot, \cdot \rangle)$. Prove that an arbitrary vector $x \in V$ can be decomposed uniquely into the form $x = u + z$, where $u \in W$ and $z \in W^\perp$. (9%) 

5. (20%) (a) Prove that the eigenvalues of a self-adjoint operator are all real. (7%) 
   (b) Show that all eigenvalues of anti-self-adjoint $T^* = -T$ operators are purely imaginary (i.e. $\sqrt{-\lambda} r \in \mathbb{R}$). (4%) 
   (c) Determine all the operators $T \in \mathcal{L}(V)$ with $T^3 = T$, $T^* = -T$. What can $T$ be? Write down your argument. (9%) 

6. (25%) Let $(V, \langle \cdot, \cdot \rangle)$ be a finite dimensional inner product vector space over $\mathbb{R}$. 
   (a) Prove that a linear functional $f \in \mathcal{L}(V, \mathbb{R})$ can always be written as $f(x) = \langle x, v \rangle$ for some $v \in V$. (14%) 
   (b) Let $\beta = \{v_1, v_2, \cdots, v_n\}$ be an orthonormal basis of $(V, \langle \cdot, \cdot \rangle)$. Prove that $\langle x, y \rangle = \langle x , [y]_{\beta} \rangle$ for all $x, y \in V$. (6%) 
   For an $x \in V$, $[x]_{\beta}$ means the column vector of coordinates relative to $\beta$. 
   (c) Let $T : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ be defined by $T(a, b, c, d) = (a + b, c + d, a - c, a + b + c + d)$. Please find $T^* : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ explicitly. (5%)
7. (15%) (a) Let $W_1, W_2$ be two finite dimensional vector subspaces of $V$. Prove that $\dim(W_1) + \dim(W_2) = \dim(W_1 \cap W_2) + \dim(W_1 + W_2)$. (9%)

(b) Let $\beta_1, \beta_2$ be bases of $W_1$ and $W_2$, respectively. Show that when $W_1 \cap W_2 = \{0\}$, $\beta_1 \cup \beta_2$ is a basis of $W_1 + W_2$. (6%)

8. (15%) Let $V$ be a finite dimensional vector space over $\mathbb{R}$.

(a) Prove that when $W$ is a $T$ invariant subspace, extend a basis $\gamma$ of $W$ to a basis $\beta$ of $V$. Prove that $[T]_{\beta}$ is of the following form, (7%)

$$\begin{pmatrix}
[T] & B \\
0 & C
\end{pmatrix}$$

(b) Let $W$ be a $T$-invariant sub-space of $V$. Prove that the characteristic polynomial of $T_W$ divides the characteristic polynomial of $T \in \mathcal{L}(V)$. (8%)

9. (15%) (a) Prove that for $A \in M_{n \times n}(\mathbb{R})$, $\dim_{\mathbb{R}} \text{span}(\{I, A, A^2, \cdots\}) \leq n$. (9%)

(b) Give an $n \times n$ example that $\dim_{\mathbb{R}} \text{span}(\{I, A, A^2, \cdots\}) = n$. (6%)

10. (15%) Prove the following statement: Let $V$ and $W$ be finite dimensional vector spaces having ordered bases $\beta$ and $\gamma$, respectively and let $T \in \mathcal{L}(V, W)$. Then for all $u \in V$, we have $[T(u)]_{\gamma} = [T]_{\beta}^\gamma[u]_\beta$.

11. (15%) (a) Prove that two finite dimensional vector spaces $V$ and $W$ are isomorphic to each other if and only if $\dim(V) = \dim(W)$. (10%)

(b) Show that a linear transformation $T \in \mathcal{L}(V, W)$ cannot be onto if $\dim(V) < \dim(W)$. (5%)