Math 113, Introduction to Abstract Algebra (Kedlaya, fall 2002)
Second midterm exam, Wednesday, November 6, 2002

This is a closed-book exam. No notes, calculator, or other assistance are permitted.
There are five problems, each on a separate page, plus an extra page if you need room for
scratch work. However, please show all your work on the problem pages; you may continue on the
back if you need more space. Work on the scratch page will not be graded.

Problem 1. Give a one-sentence answer to each of the following questions. (5 points each)

(a) Define a normal subgroup of a group $G$.

(b) Describe the effect of conjugation by $(1,3)$ on a permutation of $S_n$, in terms of its cycle
description.

(c) Define a zero divisor of a commutative ring $R$.

(d) Provide an example of a ring $R$ with unity in which the set of units is not a subring of $R$. 
Problem 2. Let the group $S_4$ act on the set $X$ of ordered pairs of elements of $\{1, 2, 3, 4\}$ (not necessarily distinct).

(a) Complete the following table by determining, for each permutation shown, the number of fixed points of that permutation acting on $X$. (10 points)

<table>
<thead>
<tr>
<th>Permutation</th>
<th>Fixed points in $X$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\begin{pmatrix} 1 \end{pmatrix}$</td>
<td></td>
</tr>
<tr>
<td>$\begin{pmatrix} 1 &amp; 2 \end{pmatrix}$</td>
<td></td>
</tr>
<tr>
<td>$\begin{pmatrix} 1 &amp; 2 &amp; 3 \end{pmatrix}$</td>
<td></td>
</tr>
<tr>
<td>$\begin{pmatrix} 1 &amp; 2 &amp; (3, 4) \end{pmatrix}$</td>
<td></td>
</tr>
<tr>
<td>$\begin{pmatrix} 1 &amp; 2 &amp; 3 &amp; 4 \end{pmatrix}$</td>
<td></td>
</tr>
</tbody>
</table>

(b) Use Burnside's formula to compute the number of orbits of the action of $S_4$ on $X$. (Hint: the sizes of the conjugacy classes of $S_4$, in the order listed in the table, are 1, 6, 8, 3, 6.) (5 points)
Problem 3.

(a) For each prime \( p \) dividing the order of the alternating group \( A_5 \), list one Sylow \( p \)-subgroup of \( A_5 \). (10 points)

(b) For each prime \( p \) in (a), determine which possibilities for the number of Sylow \( p \)-subgroups contained in \( A_5 \) are consistent with the third Sylow theorem. (10 points)
Problem 4. Let $G$ be the group $\mathbb{Z}_{10} \times \mathbb{Z}_{12}$.

(a) Find a subgroup $H$ of $G$ such that $G/H$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. (5 points)

(b) Find the order of the subgroup of $G$ generated by $(5, 5)$. (5 points)

(c) Determine the number of homomorphisms $\phi$ from $G$ to $\mathbb{Z}_2$. (Hint: any homomorphism $\phi$ is determined by its values on a set of generators of $G$.) (5 points)
Problem 5. Give careful proofs of the following statements. (10 points each)

(a) If $\phi$ is a homomorphism from $G$ to $G'$ and $H$ is a subgroup of $G$, then the image $\phi[H]$ of $H$ is a subgroup of $G'$.

(b) If a group $G$ of order 27 acts on a set $X$ of 32 elements, there must be at least one element of $X$ fixed by all of $G$.

(c) If $p$ is an odd prime number, then either $2^{(p-1)/2} + 1$ or $2^{(p-1)/2} - 1$ is a multiple of $p$. 