1. (27 points, 9 points each.) Find the following. If the answer to a question is a set, you should give it by listing or describing its elements in set brackets, \{ ... \}.

(a) The kernel of the homomorphism from \( \mathbb{Z} \) to \( D_{10} \) (the group of symmetries of a pentagon) taking each \( n \in \mathbb{Z} \) to rotation by \( n(4\pi/5) \) radians.

(b) The coset of \( A_3 \) in \( S_3 \) that contains \( (1 \ 2) \).

(c) The number of fixed points of \( \sigma^3 \), if \( \sigma \) is an element of \( S_n \) whose complete cycle decomposition consists of \( a \) cycles of length 3, \( b \) cycles of length 2, and \( n - 3a - 2b \) cycles of length 1.

2. (36 points; 9 points each.) For each of the items listed below, either give an example, or give a brief reason why no example exists. (If you give an example, you do not have to prove that it has the property stated. Examples should be specific for full credit; i.e., even if there are many objects of a given sort, you should name one.)

(a) A simple non-cyclic group.

(b) A subgroup of \( \mathbb{Z} \times \mathbb{Z} \) that is not normal.

(c) An injective (i.e., one-to-one) homomorphism \( f: \mathbb{Z} \to \mathbb{R}^\times \). (Recall that \( \mathbb{R}^\times \) denotes the group of nonzero real numbers under multiplication.)

(d) A group \( G \) and a subgroup \( H \), such that \( H \) is not the kernel of any homomorphism with domain \( G \).

3. (14 points.) Let \( G \) and \( H \) be groups, \( f: G \to H \) an injective (i.e., one-to-one) homomorphism, and \( g \in G \) an element of finite order \( n \). Show that \( f(g) \) also has order \( n \).

4. (14 points.) Let \( G \) be a group. Recall that \( Z(G) \), the center of \( G \), means \( \{ z \in G : \forall g \in G, \ zg = gz \} \). Show that \( Z(G) \) is a subgroup of \( G \). (Rotman describes this as "easy to see". I am asking you to supply the details.)

5. (9 points.) Let \( G \) be a group which acts on a set \( X \), and let \( x, y \in X \). Show that if \( \mathcal{O}(x) \) and \( \mathcal{O}(y) \) have an element in common, then they are equal. (Recall that \( \mathcal{O}(x) \) denotes \( \{ gx : g \in G \} \). The result you are to prove is part of a result proved by Rotman, that \( X \) is the disjoint union of the orbits. Hence you may not call on that result in proving this.)