Math 128a Midterm Exam
Oct 10, 2002  K.Hare

NAME (printed) :

(Family Name)  (First Name)

Signature :

Student Number :

(1) Do NOT open this test booklet until told to do so
(2) Do ALL your work in this test booklet
(3) SHOW ALL YOUR WORK
(4) CHECK THAT THERE ARE 6 PROBLEMS
(5) NO CALCULATORS
(6) No pushing, biting, or hitting

<table>
<thead>
<tr>
<th>1</th>
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1 Consider the function

\[ f(x) = 2 \cos(x) - e^x \]

a: (4 pts) Prove that this function has at least one root between 0 and \( \frac{\pi}{2} \).

Notice that \( f(0) = 1 > 0 \), \( f(\pi/2) = -e^{\pi/2} < 0 \), and \( f \) is continuous. Hence by the Intermediate Value Theorem, there exists a \( c \) between 0 and \( \pi/2 \) such that \( f(c) = 0 \), which is the desired root.

b: (3 pts) The root of \( f(x) \) is actually between 0 and 1. Using the Bisection method, how many steps would it take to determine this root between 0 and 1 to an accuracy of 10\(^{-3}\)?

We want \( \frac{1-0}{2^n} \leq 10^{-3} \) which is equivalent to \( 2^n \geq 1000 \), or \( n \geq 10 \). So we would need 10 steps of the Bisection method.
c: (3 pts) The calculation of

$$\delta - \sqrt{\delta^2 - 1}$$

is unstable for large $\delta$ due to round-off error. Suggest how to rewrite this equation to get a more accurate answer. (Justify your answer.)

Consider

$$\delta - \sqrt{\delta^2 - 1} = (\delta - \sqrt{\delta^2 - 1}) \frac{\delta + \sqrt{\delta^2 - 1}}{\delta + \sqrt{\delta^2 - 1}}$$

$$= \frac{\delta^2 - \delta^2 + 1}{\delta + \sqrt{\delta^2 - 1}}$$

$$= \frac{1}{\delta + \sqrt{\delta^2 - 1}}$$

This new equivalent formula is more stable, as you are not deleting two nearly equal numbers.
2 a: (3 pts) Define what it means for a sequence $\{p_n\}_{n=0}^{\infty}$ to converge quadratically to $p$.

We say that $p_n$ converges quadratically to $p$ if

$$\lim_{n \to \infty} \frac{|p_{n+1} - p|}{|p_n - p|^2} = \lambda$$

for some $\lambda \neq 0$.

b: (3 pts) Under what conditions does Newton’s method converge quadratically?

Newton’s method converges quadratically for a function $f$ if

- $f'(p) \neq 0$
- $f$ is continuous, and has continuous first and second derivatives.
- We start sufficiently close to the root.
c: (4 pts) Let \( p_n = \frac{1}{10^n} \). What order of convergence does \( p_n \) have?
(Justify your answer.)

*This converges quadratically. First note, \( p_n \to 0 \).*

\[
\lim_{n \to \infty} \frac{|p_{n+1} - p|}{|p_n - p|^2} = \lim_{n \to \infty} \frac{|10^{2n+1}|}{10^{2n}^2} = \lim_{n \to \infty} \frac{10^{2n+1}}{(10^n)^2} = \lim_{n \to \infty} \frac{10^n}{10^{2n+1}} = \lim_{n \to \infty} 1 = 1
\]
3: Consider the function
\[ f(x) = \frac{x + 1}{2} \]

a: (4 pts) Show that \( f(x) \) has a unique fixed point \( p \). Find \( p \). Show that the fixed point method converges to \( p \), for all starting points \( p_0 \).

Notice that \( f(1) = \frac{1+1}{2} = 1 \), so \( p = 1 \) is a fixed point. Consider any interval \([a, b]\) where \( a < 1 < b \). We see that \( f(x) \in [a, b] \) for all \( x \in [a, b] \), because \( \frac{a+1}{2} > a \) and \( \frac{b+1}{2} < b \). Further notice that \( f'(x) = 1/2 \) for all \( x \in [a, b] \). Hence the interval \([a, b]\) has exactly one fixed point, and the fixed point method will converge to this fixed point for all \( p_0 \in [a, b] \). Because \( a \) and \( b \) are arbitrary, we have that \( f(x) \) has exactly one fixed point in the real numbers, and that the fixed point method converges for all starting points \( p_0 \).
b: (3 pts) Compute $p_1$, $p_2$ and general $p_n$ of the fixed point iteration, given that $p_0 = 0$.

\[
\begin{align*}
  p_0 &= 0 \\
  p_1 &= \frac{1}{2} \\
  p_2 &= \frac{3}{4} \\
  p_n &= 1 - \frac{1}{2^n}
\end{align*}
\]

c: (3 pts) Compute $\hat{p}_0$.

\[
\begin{align*}
  \hat{p}_0 &= p_0 - \frac{(p_1 - p_0)^2}{p_2 - 2p_1 + p_0} \\
          &= 0 - \frac{(1/2 - 0)^2}{3/4 - 2(1/2) + 0} \\
          &= \frac{1/4}{-1/4} \\
          &= 1
\end{align*}
\]
4 a: (3 pts) Assume that a computer system can solve a random 1000 \times 1000 linear system in 3 seconds. How long would you expect the computer system to take to solve a 3000 \times 3000 linear system?

We know that a \( n \times n \) linear system will take \( O(n^3) \) time to solve. Thus, if we increase \( n \) from 1000 to 3000, we are tripling the size of \( n \). Hence the time expected would be \( 3^3 \times 3 \) seconds, or 81 seconds.

b: (2 pts) Assume that a computer system can solve a random 1000 \times 1000 tridiagonal system in 3 seconds. How long would you expect the computer system to take to solve a 3000 \times 3000 tridiagonal system?

We know that a \( n \times n \) tridiagonal system will take \( O(n) \) time to solve. Thus, if we increase \( n \) from 1000 to 3000, we are tripling the size of \( n \). Hence the time expected would be \( 3 \times 3 \) seconds, or 9 seconds.
c: (4 pts) Consider

\[
A = \begin{bmatrix}
2 & 4 & -2 \\
4 & 7 & -7 \\
-2 & -7 & -3
\end{bmatrix}
\]

Give a \(LDL^T\) factorization of \(A\). (Please note, in an \(LDL^T\) factorization, the diagonal entries of the \(L\) must be 1)

\[
\begin{bmatrix}
2 & 4 & -2 \\
4 & 7 & -7 \\
-2 & -7 & -3
\end{bmatrix}
= \begin{bmatrix}
1 & 0 & 0 \\
2 & 1 & 0 \\
-1 & 3 & 1
\end{bmatrix}
\begin{bmatrix}
2 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 4
\end{bmatrix}
\begin{bmatrix}
1 & 2 & 1 \\
0 & 1 & 3 \\
0 & 0 & 1
\end{bmatrix}
\]

c: (1 pt) Is \(A\) positive definite. Why or why not?

No it is not positive definite. Firstly, the diagonal of a positive definite matrix is positive, \(A\) contains a -3. Secondly, the entries of \(D\) in a \(LDL^T\) must also be positive, were as here \(D\) contains a -1. Lastly, the determinant of the leading principal matrices must all be positive, where as the determinant of \(\begin{bmatrix}
2 & 4 \\
4 & 7
\end{bmatrix}\) is -2.
5 a: (5 pts) Consider

\[ A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix} \]

Use Gaussian elimination, with partial pivoting to compute the determinant of \( A \).

*We notice that after pivoting, we get*

\[ \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \]

*Performing Gaussian elimination on this gives*

\[
\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 1 & 1 \\ 0 & 3/2 & 1/2 \\ 0 & 1/2 & 3/2 \end{bmatrix} \\
\rightarrow \begin{bmatrix} 2 & 1 & 1 \\ 0 & 3/2 & 1/2 \\ 0 & 0 & 4/3 \end{bmatrix} =: \hat{A}
\]

*So the determinant of \( \hat{A} \) is 4. As there was one row interchange, the determinant of \( A \) is -4.*
b: (5 pts) Consider the function \( f(x) \). Use the information below about \( f(x) \), and the initial guesses \( x_0 = 1, x_1 = 2 \) to compute \( x_3 \) and \( f(x_3) \) using the Secant method.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
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<tbody>
<tr>
<td>1.0</td>
<td>-1</td>
</tr>
<tr>
<td>1.1</td>
<td>(-\frac{89}{100})</td>
</tr>
<tr>
<td>1.2</td>
<td>(-\frac{26}{100})</td>
</tr>
<tr>
<td>1.3</td>
<td>(-\frac{6}{10})</td>
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<tr>
<td>1.4</td>
<td>(-\frac{11}{2})</td>
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<td>1.5</td>
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<tr>
<td>1.6</td>
<td>(-\frac{13}{100})</td>
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<tr>
<td>1.7</td>
<td>(-\frac{11}{100})</td>
</tr>
<tr>
<td>1.8</td>
<td>(-\frac{7}{100})</td>
</tr>
<tr>
<td>1.9</td>
<td>(-\frac{1}{100})</td>
</tr>
<tr>
<td>2.0</td>
<td>1</td>
</tr>
</tbody>
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\[
x_2 = x_1 - \frac{f(x_1)(x_1 - x_0)}{f(x_1) - f(x_0)}
\]

\[
= 2 - \frac{1(2 - 1)}{1 - (-1)}
\]

\[
= 2 - \frac{1}{2}
\]

\[
= \frac{3}{2}
\]

\[
x_3 = x_2 - \frac{f(x_2)(x_2 - x_1)}{f(x_2) - f(x_1)}
\]

\[
= 3 - \frac{-1/4(3/2 - 2)}{2 - (-1/4 - 1)}
\]

\[
= 3 - \frac{1/8}{2 - 5/4}
\]

\[
= 3 - \frac{1}{2 + 10}
\]

\[
= \frac{8}{5}
\]

\[
f(x_3) = \frac{-1}{25}
\]
6 a: (2 pts) Define what a diagonally dominate matrix is.

A strictly diagonally dominant matrix $A$ is such that

$$|a_{i,i}| > \sum_{j \neq i} |a_{i,j}|$$

for all rows $i$.

b: (3 pts) Prove or find a counter example. The matrix $A$ is a diagonally dominate matrix if and only if $A^T$ is.

Consider the matrix

$$A = \begin{bmatrix} 1 & 0 \\ 10 & 100 \end{bmatrix}$$

Clearly $A$ is strictly diagonally dominate, and $A^T$ is not.
c: (2 pts) Define what a permutation matrix is.

A permutation matrix $A$ is an $n \times n$ matrix with exactly one 1 in each row and one 1 in each column. All other entries are 0.

d: (3 pts) Prove or find a counter example. The matrix $A$ is a permutation matrix if and only if $A^T$ is.

If $A$ is permutation matrix then $A$ has exactly one 1 in each row, and hence $A^T$ has exactly one 1 in each column. If $A$ is permutation matrix then $A$ has exactly one 1 in each column, and hence $A^T$ has exactly one 1 in each row. If $A$ is a permutation matrix, then all other entries are 0, and hence in $A^T$, all other entries are 0. Hence if $A$ is a permutation matrix, then $A^T$ is a permutation matrix.

Further if $A^T$ is a permutation matrix, then $(A^T)^T = A$ is a permutation matrix.