MAT 110 - MIDTERM 9/27/02

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1. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ the linear transformation defined by

$$T(x, y, z) = (x + y + z, x + 3y + 5z).$$

a) Find $N(T), R(T)$ and compute $\dim N(T), \dim R(T)$.

b) Let $\beta$ and $\gamma$ the standard bases for $\mathbb{R}^3$ and $\mathbb{R}^2$ respectively. Consider also $\alpha = \{(1, 1, 1), (2, 3, 4), (3, 4, 6)\}$ basis for $\mathbb{R}^3$.

Compute $Q$ the change of coordinate matrix from $\beta$ to $\alpha$ and the representation matrices $[T]_{\alpha}^\gamma, [T]_{\beta}^\gamma$.

Check that

$$[T]_{\alpha}^\gamma \cdot Q = [T]_{\beta}^\gamma.$$

2. Let $V$ and $W$ two vector spaces over the rational numbers field $\mathbb{Q}$ and $T : V \to W$ which satisfies

$$T(x + y) = T(x) + T(y).$$

Prove that $T$ is a linear transformation.

3. Let $m$ and $n$ two positive integers. Construct a linear transformation $T$ such that $\text{nullity}(T) = m$ and $\text{rank}(T) = n$.

4. Let $T : V \to V$ a linear transformation, where $V$ is a finite-dimensional vector space. Prove that if $\text{rank}(T) = \text{rank}(T^2)$ then $R(T) \cap N(T) = \{0\}$.

5. Let $A, B$ two square matrices, $A, B \in M_{n \times n}(\mathbb{R})$ such that $I_n - AB$ is invertible. Prove that $I_n - BA$ is invertible.