1. (12 points, 4 points each.) Complete the following definitions.

(a) If \( X \) is a set, then a permutation of \( X \) means ...

(b) If \( R \) and \( S \) are commutative rings, then a map \( f: R \rightarrow S \) is called a homomorphism of commutative rings if ...

(c) A proper ideal \( I \) of a commutative ring \( R \) is said to be a maximal ideal if ...

2. (36 points; 4 points each.) For each of the items listed below, either give an example, or give a brief reason why no example exists. (If you give an example, you do not have to prove that it has the property stated. Examples should be specific for full credit; i.e., even if there are many objects of a given sort, you should name one.)

(a) An element \( \sigma \in S_6 \) such that \( \sigma(1 \ 2 \ 3) \sigma^{-1} = (2 \ 4 \ 6) \).

(b) An injective (i.e., one-to-one) homomorphism \( f: \mathbb{Z} \rightarrow \mathbb{R}^x \). (Recall that \( \mathbb{R}^x \) denotes the group of nonzero real numbers under multiplication.)

(c) A factorization of the polynomial \( 3x^3 + 29x^2 - 4x - 2 \) as a product of two polynomials of lower degree in \( \mathbb{Q}[x] \).

(d) A ring \( R \), an ideal \( I \subseteq R \), and elements \( a \neq b \) of \( R \) such that \( a + I = b + I \).

(e) A polynomial \( f(x) \in \mathbb{Q}[x] \) which has no root in \( \mathbb{Q} \), but which is reducible in \( \mathbb{Q}[x] \).

(f) A field with exactly 100 elements.

(g) An ideal \( I \subseteq \mathbb{Z}[x] \) such that \( \mathbb{Z}[x]/I \) is isomorphic to \( \mathbb{Z}[i] \), the ring of Gaussian integers.

(h) Two elements of \( \mathbb{Z}_5[x] \) that are associates, but are not equal.

(i) A unique factorization domain which is not a principal ideal domain.

3. Short proofs. (22 points = 6 + 8 + 8.)

(a) If \( f: X \rightarrow Y \) and \( g: Y \rightarrow Z \) are set maps such that the composite map \( g \circ f: X \rightarrow Z \) is injective, show that \( f \) is injective.

(b) Suppose a group \( G \) acts on a set \( X \), and let \( S = \{ \sigma \in G \mid \forall x \in X, \sigma x = x \} \). Show that \( S \) is a normal subgroup of \( G \). (You must show both that it is a subgroup and that it is normal.)

(c) Suppose \( P_1 \geq P_2 \geq \ldots \geq P_n \geq P_{n+1} \geq \ldots \) is a decreasing sequence of prime ideals. Show that the ideal \( I = \bigcap_{n \geq 1} P_n \) is prime. (Here you are to take for granted that \( I \) is an ideal, in contrast to the homework problem this is taken from, where you had to prove both that it was an ideal and that it was prime.)