Fall 2001 Second Midterm R. Borcherds
Math 121AMidterm, Tuesday October 30, 9:40-11:00.

Please make sure that your name is on everything you hand in.
You are allowed calculators and 1 page of notes.
All questions have about the same number of marks.

1. Find the point on the sphere $x^2 + y^2 + z^2 = 1$ for which $x + 2y + 2z$ is a maximum.

2. If $w = \int_{x}^{x^2} e^{-u^2} du$ find $dw/dx$.

3. Show that $x/(x^2+y^2)$ is harmonic (in other words, it satisfies Laplace's equation). Find an analytic function $f(z) = f(x + iy)$ of which it is the real part. (Hint: the function has degree $-1$, so the same is likely to be true for $f$.) Find the conjugate harmonic function $Im(f(x + iy))$.

4. Evaluate the contour integral $\int_{C} e^{z}dz/(z-2)$ if $C$ is the circle of center 0 and radius 3.

5. Find the Laurent series for $\sin(\pi z)/(4z^2 - 1)$ about the point $z = 1/2$ and use this to find the residue at $z = 1/2$.

6. Evaluate the integral
$$\int_{-\infty}^{\infty} \frac{\cos(x)}{x^2 + 1} dx$$
(Hint: $\cos(x) = Re(exp(ix))$.)

7. Evaluate the integral
$$\int_{0}^{2\pi} \frac{d\theta}{5 - 3 \cos(\theta)}$$
(Hint: put $z = e^{i\theta}$.)

8. A flat plate is in the shape of a quarter circle of radius 1, and consists of the points $(x, y)$ in the plane with $x \geq 0$, $y \geq 0$, $x^2 + y^2 \leq 1$. The curved part of the boundary is insulated and the edges $y = 0$ and $x = 0$ are held at temperatures of 0° and 100°. Find the temperature distribution $T(x, y)$ inside the plate. (Hint: if the quarter circle is regarded as part of the unit circle in the $z$-plane, then the mapping function $w = \log(z)$ maps the quarter circle to an infinitely long rectangle in the $w$-plane.)