1. (32 points, 8 points apiece) Complete the following definitions. In defining any term, you may use terms defined before it in the text.

(a) If \( \varphi : G \to H \) is a homomorphism of groups, then the kernel of \( \varphi \) (denoted \( \ker(\varphi) \)) means

(b) If \( G \) is a group, then the center of \( G \) (denoted \( Z(G) \)) means

(c) If \( V \) is a vector space, then a basis of \( V \) means

(d) If \( F \) is a field and \( f(x), g(x) \) are nonzero polynomials over \( F \), then the greatest common divisor of \( f(x) \) and \( g(x) \) (denoted \( \gcd(f(x), g(x)) \)) means the unique monic polynomial \( a(x) \) such that

2. (36 points: 9 points each.) For each of the items listed below, either give an example, or give a brief reason why no example exists. (If you give an example, you do not have to prove that it has the property stated.)

(a) A polynomial \( f(x) \in \mathbb{Q}[x] \) which is reducible over \( \mathbb{Q} \), but has no root in \( \mathbb{Q} \).

(b) A polynomial \( f(x) \in \mathbb{Z}[x] \) which has a root in \( \mathbb{Q} \) but no root in \( \mathbb{Z} \).

(c) Two nonisomorphic abelian groups of order 10.

(d) Groups \( G \) and \( H \), a homomorphism \( \varphi : G \to H \), and an element \( g \in G \) such that \( g \) has infinite order, and \( \varphi(g) \) has order 3.

3. (12 points) Suppose \( G \) is a group which acts on a set \( S \), and \( s \) is an element of \( S \). Recall that \( G_s \) denotes \( \{ g \in G \mid g \cdot s = s \} \).

Prove that for any \( a \in G \) one has \( G_{as} = aG_s a^{-1} \).

4. (20 points) Let \( G \) be a group and \( N \) a normal subgroup of \( G \). Show that \( G/N \) is abelian if and only if for all \( a, b \in G \), one has \( aba^{-1}b^{-1} \in N \).

(If you correctly derive a necessary and sufficient condition close to this, but do not succeed in transforming it to precisely the above form, you will get appropriate partial credit.)