Prof. Bjorn Poonen
December 4, 2001

MATH 55 PRACTICE FINAL

Do not write your answers on this sheet. Instead please write your name, your student ID, your TA's name, your section time, "practice," and all your answers in your blue books. IMPORTANT: Write your answers to problems 1–10 on the first page or two of your blue book, without the calculations you did to get the answers. To guarantee full credit and to qualify for partial credit, you should show your work on *later* pages of the blue book. Total: 200 pts., 170 minutes.

(1) For each of (a)-(d) below: If the statement is true (always), write TRUE. Otherwise write FALSE. (Please do not use the abbreviations T and F, since in handwriting they are sometimes indistiguishable.) No explanations are required in this problem.

(a) If $f : S \to T$ is a function and $A \subseteq B \subseteq S$, then $f(B - A) = f(B) - f(A)$.

(b) If f and g are independent random variables defined on the same sample space, then the standard deviation $\sigma(f + g)$ of $f + g$ equals $\sqrt{(\sigma(f))^2 + (\sigma(g))^2}$.

(c) There exists an injection from the set of rational numbers to the set of integers.

(d) The program

procedure *fibonacci(n*: nonnegative integer)
if $n = 0$ **then** *fibonacci*(0) := 0
else if $n = 1$ **then** *fibonacci*(1) := 1
else *fibonacci*(n) := *fibonacci*($n - 1$) + *fibonacci*($n - 2$)

performs a total of $\Theta(n)$ additions to compute the n^{th} Fibonacci number.

(2) How many elements of $\{1, 2, \ldots, 10\} \times \{1, 2, \ldots, 10\}$ must be taken in order to guarantee that there are two elements, say (i, j) and (i', j'), such that $i + j = i' + j'$?

(3) How many triples (x, y, z) of integers satisfy $x \geq 2$, $y \geq 3$, $0 \leq z \leq 8$, and $x + y + z = 14$?

(4) Express in the closed form the generating function for the sequence a_0, a_1, ..., where $a_n = \binom{n}{2}$.

(5) A sequence $a_0, a_1, \ldots$ satisfies $a_0 = 1$ and

$$a_n = 2a_{n-1} + n^2 + 5^n$$

for $n \geq 1$. Find an explicit formula for a_n.

(6) What is the last digit when 5^{100} is written in base 7?

(7) For how many positive integers n less than or equal to 105 is it true that $n \bmod 3 = 1$, or $n \bmod 5 = 1$, or $n \bmod 7 = 1$?

(8)
(a) How many functions are there from $\{1,2,3\}$ to $\{1,2,3,4\}$?
(b) How many of the functions in part (a) are injective?
(c) How many of the functions in part (a) are surjective?
(d) How many of the functions in part (a) have the property that $|f(\{1,2\})| = 2$?

(9) A 10-element subset of $\{1, 2, \ldots, 40\}$ is chosen at random, with each 10-element subset being equally likely. What is the expected number of pairs $(i, i+1)$ of consecutive integers such that i and $i+1$ both belong to the random subset?

(10) A *nonempty* subset of $\{1,2,3,4,5\}$ is chosen at random, with each nonempty subset being equally likely. For any such subset A, let $f(A)$ be the largest element of A, and let $g(A) = |A|$. Are the random variables f and g independent?

(11) List the permutations of $\{1,2,3\}$ in lexicographic order.

(12) Prove or disprove that for any two sets A and B contained in a universe U, $\overline{A - B} = B - A$.

(13) Prove that if a positive integer n has exactly three positive divisors, then $n = p^2$ for some prime number p.

This is the end! At this point, you may want to look over the exam to make sure you have not omitted any problems. Check that your answers make sense! Please take this exam with you as you leave.

Prof. Bjorn Poonen (solutions by J. Steever)
December 6, 2001

MATH 55 PRACTICE FINAL SOLUTIONS

(1) For each of (a)-(d) below: If the statement is true (always), write TRUE. Otherwise write FALSE. (Please do not use the abbreviations T and F, since in handwriting they are sometimes indistiguishable.) No explanations are required in this problem.

(a) If $f : S \to T$ is a function and $A \subseteq B \subseteq S$, then $f(B - A) = f(B) - f(A)$.

FALSE. Consider the constant function $f : \{1,2\} \to \{1\}$ (i.e. $f(1) = f(2) = 1$). If $B = \{1,2\}$ and $A = \{1\}$ then $f(B - A) = f(\{2\}) = 1$, however $f(B) - f(A) = \{1\} - \{1\} = \emptyset$.

(b) If f and g are independent random variables defined on the same sample space, then the standard deviation $\sigma(f+g)$ of $f+g$ equals $\sqrt{(\sigma(f))^2 + (\sigma(g))^2}$.

TRUE.

$$\begin{aligned} \sigma(f+g) &= \sqrt{V(f+g)} \\ &= \sqrt{V(f) + V(g)} \\ &= \sqrt{(\sigma(f))^2 + (\sigma(g))^2} \end{aligned}$$

(c) There exists an injection from the set of rational numbers to the set of integers.

TRUE. The set of rationals is countable, therefore there is a bijection, and hence an injection, from the set of rationals into the set of integers.

(d) The program

procedure *fibonacci(n*: nonnegative integer)
if $n = 0$ **then** *fibonacci*(0) := 0
else if $n = 1$ **then** *fibonacci*(1) := 1
else *fibonacci*(n) := *fibonacci*($n-1$) + *fibonacci*($n-2$)

performs a total of $\Theta(n)$ additions to compute the n^{th} Fibonacci number.

FALSE. Let a_n denote the number of additions required by the algorithm to produce the n^{th} Fibonacci number. Then a simple analysis of the algorithm yield the relation $a_n = a_{n-1} + a_{n-2} + 1$, with initial conditions $a_0 = 0, a_1 = 0$. Solving this relation, we get the formula

$$a_n = \frac{5-\sqrt{5}}{10} \cdot (\frac{1+\sqrt{5}}{2})^n + \frac{5+\sqrt{5}}{10} \cdot (\frac{1-\sqrt{5}}{2})^n - 1$$

which is not $\Theta(n)$.

(2) How many elements of $\{1,2,\ldots,10\} \times \{1,2,\ldots,10\}$ must be taken in order to guarantee that there are two elements, say (i,j) and (i',j'), such that $i+j = i'+j'$?

A total of 20 elements must be picked. Over all (i, j) in the set, $i + j$ ranges over the 19 natural numbers 2 thru 20. Thus, by the Pigeonhole Principle, any twenty pairs must contain at least two whose coordinate sums are equal.

(3) How many triples (x, y, z) of integers satisfy $x \geq 2$, $y \geq 3$, $0 \leq z \leq 8$, and $x + y + z = 14$?

Let $u + 2 = x$ and $v + 3 - y$. Then the above question is equivalent to the number of triples (u, v, z) of integers satisfying $u \geq 0$, $v \geq 0$, $0 \leq z \leq 8$, and $u + v + z = 9$. There are $\binom{3+9-1}{9}$ ways for natural numbers u, v, z to satisfy $u + v + z = 9$. Furthermore, there is only one such triple with $z > 8$, $(0, 0, 9)$. Thus the answer is $\binom{11}{9} - 1 = 54$.

(4) Express in the closed form the generating function for the sequence a_0, a_1, ..., where $a_n = \binom{n}{2}$.

$$\begin{aligned}
\sum_{n=0}^{\infty} \binom{n}{2} x^n &= 0 + 0x + \sum_{n=2}^{\infty} \binom{n}{2} x^n \\
&= \sum_{k=0}^{\infty} \binom{k+2}{2} x^{k+2} (\mathrm{k} + 2 = \mathrm{n}) \\
&= x^2 (\sum_{k=0}^{\infty} \binom{k+2}{2} x^k \\
&= x^2/(1-x)^3
\end{aligned}$$

(5) A sequence $a_0, a_1, \ldots$ satisfies $a_0 = 1$ and

$$a_n = 2a_{n-1} + n^2 + 5^n$$

for $n \geq 1$. Find an explicit formula for a_n.

The solution to this problem occurs in three steps. First, we find the general solution to the associated homogeneous relation $a_n = 2a_{n-1}$. Next we find a particular solution to the nonhomogeneous relation $b_n = 2b_{n-1} + n^2$. Finally, we find a particular solution to the nonhomogeneous relation $c_n = c_{n-1} + 5^n$. The purpose or this method is that if a_n, b_n, and c_n are three such sequences, then $a_n + b_n + c_n = 2a_{n-1} + 2b_{n-1} + n^2 + 2c_{n-1} + 5^n = 2(a_{n-1} + b_{n-1} + c_{n-1}) + n^2 + 5^n$. Thus, the sequence $\{a_n + b_n + c_n\}$ is a solution to the required relation. Furthermore, any such solution can be shown to be of this form.

To solve the homogeneous relation $a_n = 2a_{n-1}$, we examine its characteristic equation $r - 2 = 0$. Thus, the general solution to this relation is $a_n = \alpha 2^n$.

From the text, we know that the relation $b_n = 2b_{n-1} + n^2$ has a particular solution of the form $b_n = \beta_2 n^2 + \beta_1 n + \beta_0$. Plugging this formula into the relation, we find

$$(\beta_2 + 1)n^2 + (\beta_1 - 4\beta_2)n + (\beta_0 + 2\beta_2 - 2\beta_1) = 0$$

. Thus, $\beta_2 = -1, \beta_1 = -4, \beta_0 = -6$, or $b_n = -(n^2 + 4n + 6)$.

Finally, from the text, we know that the relation $c_n = 2c_{n-1} + 5^n$ has a solution of the form $c_n = \gamma 5^n$. Plugging this formula into the relation, we find $\gamma = 5/3$. So $c_n = 5^{n+1}/3$.

Thus, the general solution to the original nonhomogeneous equation is

$$a_n = \alpha 2^n - (n^2 + 4n + 6) + 5^{n+1}/3$$

. Solving for the particular solution requested (with $a_0 = 1$), we get $1 = \alpha - (6) + 5/3$, yielding $\alpha = 16/3 = 2^4/3$. Thus, our particular solution is

$$a_n = 2^{n+4}/3 - (n^2 + 4n + 6) + 5^{n+1}/3.$$

(6) What is the last digit when 5^{100} is written in base 7?

As the last digit of 5^{100} in base 7 is simply 5^{100} **mod** 7. By performing modulo 7 arithmatic, we find $5^2 \textbf{ mod } 7 = 4$. Thus, $5^3 \textbf{ mod } 7 = -1 \textbf{ mod } 7$. So

$$5^{100} \textbf{ mod } 7 = (5^3)^{33} 5 \textbf{ mod } 7 = (-1)^{33} 5 \textbf{ mod } 7 = -5 \textbf{ mod } 7 = 2.$$

(7) For how many positive integers n less than or equal to 105 is it true that $n \textbf{ mod } 3 = 1$, or $n \textbf{ mod } 5 = 1$, or $n \textbf{ mod } 7 = 1$?

Let A denote the set of positive integers n less that or equal to 5 such that $n \textbf{ mod } 3 = 1$. Let B and C denote the respective sets for **mod** 5 and **mod** 7, respectively. Then it is clear that the answer to our question is equal to $|A \cup B \cup C|$. By the principle of inclusion-exclusion,

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

Obviously, $|A| = 35$, $|B| = 21$, and $|C| = 15$. Upon inspection, if an integer n is in both A and B, then $n - 1$ must be divisible by both 3 and 5, and hence 15. Thus, $|A \cap B| = 7$. Similarly, $|A \cap C| = 5$ and $|B \cap C| = 3$. Finally, $|A \cap B \cap C| = 1$. So $|A \cup B \cup C| = 35 + 21 + 15 - 7 - 5 - 3 + 1 = 57$.

(8)

(a) How many functions are there from $\{1, 2, 3\}$ to $\{1, 2, 3, 4\}$?

Each of the 3 values in the domain can be sent to any of the 4 elements of the codomain. So there are $4^3 = 64$ such functions.

(b) How many of the functions in part (a) are injective?

There are 4 possible values for $f(1)$. Once one of these have been chosen, there are 3 choices for $f(2)$, and then 2 choices for $f(3)$. Thus, there are $4 \cdot 3 \cdot 2 = 24$ injective functions.

(c) How many of the functions in part (a) are surjective?

There are no such surjective functions as $f(\{1, 2, 3\})$ can have at most three elements.

(d) How many of the functions in part (a) have the property that $|f(\{1, 2\})| = 2$?

In order to have the necessary property, it is necessary and sufficient that f must map 1 and 2 to different numbers. Thus, in defining a function, there are 4 choices for $f(1)$, then 3 choices for $f(2)$, followed by 4 choice for $f(3)$, yielding $4 \cdot 3 \cdot 4 = 48$ such functions.

(9) A 10-element subset of $\{1, 2, \ldots, 40\}$ is chosen at random, with each 10-element subset being equally likely. What is the expected number of pairs $(i, i + 1)$ of consecutive integers such that i and $i + 1$ both belong to the random subset?

There are $\binom{40}{10}$ elements in our sample space of all 10-element subsets. For each $1 \leq i \leq 39$, let

$$f_i(A) = \begin{cases} 1 & \text{if } \{i, i+1\} \subseteq A \\ 0 & \text{otherwise} \end{cases}$$

Then $f(A) = \sum_{i=1}^{39} f_i(A)$ gives the number of pairs $(i, i+1)$ contained in S. Also, for each i, there are $\binom{38}{8}$ subsets A such that $f_i(A) = 1$. Thus, for each i,

$$E(f_i) = \sum_{A \in S} p(A) f_i(A) = \binom{38}{8} / \binom{40}{10} = 3/52.$$

Thus, $E(X) = \sum_{i=1}^{39} E(f_i) = 39 \cdot (3/52) = 9/4 = 2.25$.

(10) A *nonempty* subset of $\{1, 2, 3, 4, 5\}$ is chosen at random, with each nonempty subset being equally likely. For any such subset A, let $f(A)$ be the largest element of A, and let $g(A) = |A|$. Are the random variables f and g independent?

The random variables are not independent. There are $2^5 - 1 = 31$ possible nonempty subsets. As all of them have equal probability of being chosen, the probability of each is $1/31$. Now there is only 1 nonempty subset A such that $f(A) = 1$, namely $A = \{1\}$ and there is only one subset $B = 1, 2, 3, 4, 5$ such that $g(B) = 5$ and $A \neq B$. Thus,

$$p(f(s) = 1 and g(s) = 5) = 0 \neq (1/31)^2 = p(f(s) = 1) \cdot p(g(s) = 1).$$

(11) List the permutations of $\{1, 2, 3\}$ in lexicographic order.

$(1, 2, 3), (1, 3, 2), (2, 1, 3), (2, 3, 1), (3, 1, 2), (3, 2, 1)$

(12) Prove or disprove that for any two sets A and B contained in a universe U, $\overline{A - B} = B - A$.

False. Let $U = \{1, 2, 3\}$, $A = \{1\}$, and $B = \{2\}$. Then $B - A = \{2\}$. However, $\overline{A - B} = \overline{\{1\}} = \{2, 3\}$.

(13) Prove that if a positive integer n has exactly three positive divisors, then $n = p^2$ for some prime number p.

As both n and 1 divide n, and, clearly, $n \neq 1$, the 3 divisors of n must be of the form $1 < p < n$, where p is a positive integer. If p is not prime, then it has a prime divisor $q < p$ which would then divide n, contradicting n having only three divisors. Thus, p is prime. Now we know that $n = mp$ for some positive integer $m < n$. But m can't be 1 as $p < n$ and m is a divisor of n. So $m = p$. Thus, $n = p^2$ for some prime p.