1. (28 points, 7 points apiece) Complete each of the following definitions. (Do not give examples or other additional facts about the concepts defined.)

(a) A **metric space** is

(b) A set $X$ is said to be **countable** if

(c) The **radius of convergence** $R$ of the power series $\sum_{n=0}^{\infty} c_n z^n$ is defined to be

(d) If $S$ is an ordered set, $E$ a subset of $S$, and $x$ an element of $S$, then we call $x$ the **least upper bound** of $E$, and write $x = \sup E$, if

2. (40 points; 10 points each.) For each of the items listed below, either give an example, or give a brief reason why no example exists. (If you give an example, you do not have to prove that it has the property stated.)

(a) A non-convergent Cauchy sequence $(a_n)$ in a metric space $X$.

(b) A sequence of points in the interval $[0,1] \subseteq R$ having no convergent subsequence.

(c) A convergent series $\sum a_n$ such that $\lim_{n \to \infty} |a_{n+1}/a_n| = 1$.

(d) A family of subsets $G_\alpha \subseteq R$ (where $\alpha$ ranges over some set $A$) such that every finite subfamily $\{G_{\alpha_1}, \ldots, G_{\alpha_n}\}$ has nonempty intersection, but the whole family has empty intersection, $\bigcap_{\alpha \in A} G_\alpha = \emptyset$.

3. (32 points) Let $X$ be a metric space and $K$ a compact subset of $X$.

(a) (12 points) Show that for every real number $\epsilon > 0$ there exists a finite subset $S \subseteq K$ such that $(\forall x \in K)(\exists s \in S)$ $d(x, s) < \epsilon$; i.e., such that each point of $K$ is within distance $\epsilon$ of some point of $S$.

(b) (20 points) Deduce from the result of (a) that $K$ contains a finite or countable subset $T$ which is dense in $K$. (Recall that a subset of $K$ is called dense if every point of $K$ is a member of the subset or a limit point of the subset. Suggestion: apply part (a) to a countable sequence of values of $\epsilon$ which approach 0.) In doing this part you may assume the result of (a) even if you did not succeed in proving it.