Birational Geometry in Characteristic $p > 0$

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In my previous talk, I discussed the birational geometry of varieties defined over the complex numbers.

In this talk I will focus on varieties defined over an algebraically closed field of characteristic \( p > 0 \).

Throughout this talk \( X \subset \mathbb{P}^N_k \) will denote a \( d \)-dimensional projective variety defined over an algebraically closed field \( k = \overline{k} \) of characteristic \( p > 0 \).

Typically we assume that \( X \) is smooth or has mild singularities.

We begin by recalling some of the highlights from the MMP in characteristic 0 that we will use as a guiding principle in characteristic \( p > 0 \).

As usual \( \omega_X = \bigwedge^d T_X^\vee \) denotes the **canonical line bundle** so that sections \( s \in H^0(\omega_X^\otimes m) \) can be locally written as \( s|_U = f(x_1, \ldots, x_d)dx_1 \wedge \ldots \wedge dx_d \).

\[ R(\omega_X) = \bigoplus_{m \geq 0} H^0(\omega_X^\otimes m) \] is the **canonical ring**.
The canonical ring

The fundamental result of the MMP in characteristic 0 is

**Theorem (Birkar-Cascini-Hacon-McKernan, Siu)**

Let $X$ be a smooth projective variety over an algebraically closed field of characteristic 0, then $R(\omega_X)$ is finitely generated.

- Note that $R(\omega_X)$ is a birational invariant.
- The **Kodaira dimension** $\kappa(X) \in \{-1, 0, 1, \ldots, d = \dim X\}$ is given by $\kappa(X) = \text{tr.deg.}_k R(\omega_X) - 1$.
- We say that $X$ is of **general type** if $\kappa(X) = d$. In this case $X_{\text{can}} = \text{Proj}(R(\omega_X))$ is a distinguished representative of the birational class of $X$ with mild singularities such that $\omega_{X_{\text{can}}}^\otimes m$ is very ample for some $m > 0$.
- Thus $H^0(\omega_X^\otimes m)$ defines an embedding $\phi_m : X_{\text{can}} \hookrightarrow \mathbb{P}^n = \mathbb{P}H^0(\omega_X^\otimes m)$ with $\omega_{X_{\text{can}}}^\otimes m = \phi_m^* \mathcal{O}_{\mathbb{P}^n}(1)$. 

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This allows us to construct projective moduli spaces.

Eg. When $d = 1$, curves of general type correspond to curves of genus $g \geq 2$ or equivalently such that the degree of the canonical line bundle is positive $\deg(\omega_X) = 2g - 2 > 0$.

These curves are parametrized by $M_g$, an irreducible variety of $3g - 3$ which can be compactified to $M_g \subset \bar{M}_g$ in a geometrically meaningful way.

The points of $\bar{M}_g \setminus M_g$ correspond to stable curves, i.e. curves with node singularities and ample canonical line bundle.

It turns out that these results also hold over algebraically closed fields of characteristic $p > 0$.

In fact there is more good news in dimension $d = 2$. 
When \( d = 2 \), Bombieri and Mumford (1976) prove that the Enriques classification of surfaces also holds if \( \text{char}(k) = p > 0 \).

There are a few surprises, such as quasi-hyperelliptic surfaces. These have \( c_1(\omega_X) \equiv 0 \) and are fibered over an elliptic curve, but the fibers are cuspidal rational curves!

This can not happen in characteristic 0.

Even more surprisingly Ekedal (1988) showed that \( H^0(\omega_X \otimes^5) \) defines an embedding \( \phi_5 : X_{\text{can}} \hookrightarrow \mathbb{P}^N \).

He even shows that \( H^i(\omega_X \otimes^m_{\text{can}}) = 0 \) for \( i > 0, \ m \geq 2 \) except in one case (\( p = 2, \ m = 2, \ldots \)).
Theorem (Alexeev, Kollár, Patakfalvi, Hacon-Kovács and others)

Fix the volume $v = c_1(\omega_{X_{\text{can}}})^2$. Then there exists an integer $p_0$ such that for all $p > 0$ there is a projective moduli space of stable surfaces $\bar{M}_{2,v}$.

- In fact this moduli space is defined over $\mathbb{Z}[1/m]$.
- It is expected that after some technical issues are resolved, it will be defined over $\mathbb{Z}$.
- The main remaining technical issues are the minimal model program for 3-folds and $p = 2, 3$, inversion of adjunction type results and semistable reduction.
- Next I will discuss some of the technical difficulties that we encounter in positive characteristics.
Technical difficulties

- The first difficulty is that in positive characteristics resolution of singularities is only known in dimensions $\leq 3$ (Abhyankar, Cossart-Piltant, Cutkosky and others).
- Conjecturally it is expected in all dimensions and de Jong’s theory of alterations provides us with a good substitute.
- A more serious issue is the failure of vanishing theorems.
- Recall **Serre vanishing**: If $L$ is an ample line bundle and $F$ is a coherent sheaf on a projective variety, then $H^i(F \otimes L^\otimes m) = 0$ for all $m \gg 0$.
- **Kodaira vanishing** and its generalizations due to Kawamata, Viehweg and others is a much more precise statement: If $X$ is smooth, then $H^i(\omega_X \otimes L) = 0$ for $i > 0$.
- By Kawamata and Viehweg we may even assume that $X$ has some mild (klt) singularities and $L$ is nef and big (instead of ample). This formulation is preferable for birational geometry.
- It is well known that Kodaira vanishing in characteristic $p > 0$ fails as soon as $d \geq 2$ (Raynaud, Lauritzen-Rao and others).
Why is Kodaira vanishing useful?

Consider $S \subset X$ a smooth divisor in a smooth variety and $L$ an ample line bundle.

There is a short exact sequence

$$0 \to \omega_X \otimes L \to \omega_X(S) \otimes L \to \omega_S \otimes L|_S \to 0.$$ 

Kodaira vanishing implies that

$$H^0(\omega_X(S) \otimes L) \to H^0(\omega_S \otimes L|_S)$$

is surjective.

Therefore we can deduce results on the geometry of $X$ from results on the geometry of $S$.

This allows for proofs by induction on the dimension.

For example if $S \sim K_X$ is ample and $\omega_S^{\otimes k}$ is base point free, then $\omega_X^{\otimes 2k}$ is base point free.

Proof: Clearly the base locus is contained in $S$ and we conclude $\omega_X^{\otimes 2k} \cong (\omega_X(S))^{\otimes k}$ and

$$H^0((\omega_X(S))^{\otimes k}) \to H^0(\omega_S^{\otimes k})$$

is surjective.
Serre vanishing and the Frobenius

To remedy the failure of Kodaira vanish we combine Serre vanishing with the Frobenius morphism.

Let $F : X \to X$ be the morphism defined by $F^*(f) = f^p$.

Note that $(f + g)^p = f^p + g^p$ and $(fg)^p = f^p g^p$ (since $\text{char}(k) = p$) and so we have a ring homomorphism $\mathcal{O}_X \to F_* \mathcal{O}_X$.

It is easy to see that if $L$ is a line bundle then $F^* L \cong L^p$ and so by the projection formula $(F^e_\omega X) \otimes L \cong F^e_\omega (\omega X \otimes L^pe)$. 

By Grothendieck duality applied to $\mathcal{O}_X \to F^e_\mathcal{O}_X$ we have a trace map $tr : F^e_\omega X \to \omega X$ which can be described locally $tr(x^{p(j+1)-1}dx) = x^j dx$.

The trace map is compatible with adjunction $\omega_X(S) \to \omega_S$.

Combining this, we have a commutative diagram:
\[ W_x(S) \otimes L \to W_s \otimes L|_S \]

\[
\begin{array}{c}
\uparrow^{t_n^e} \\
F^e \big( W_x(S) \otimes L^p^e \big) \to F^e \big( W_s \otimes L^p^e \big)
\end{array}
\]

\[ 0 = H^2 \big( W_x \otimes L^p^e \big) = H^1 \big( F^e \big( W_x \otimes L^p^e \big) \big) \]

by Some Vanishing

\[
\begin{array}{c}
\uparrow^{t_n^e} \\
H^0 \big( W_x(S) \otimes L \big) \to H^0 \big( W_s \otimes L|_S \big)
\end{array}
\]

\[
\begin{array}{c}
\uparrow^{H^0(t_n^e)} \\
H^0 \big( W_x(S) \otimes L^p^e \big) \to H^0 \big( W_s \otimes L^p^e \big)
\end{array}
\]

Let \[ s^0 \big( W_s \otimes L|_S \big) = \text{Im} \big( H^0(t_n^e) \big) \]

then

\[ H^0 \big( W_x(S) \otimes L \big) \to s^0 \big( W_x \otimes L|_S \big) \]
The challenge is then to identify the space of **Frobenius stable sections** $S^0(\omega_S \otimes L) \subset H^0(\omega_S \otimes L)$.

When $L$ is sufficiently ample (and $S$ is smooth), then $S^0(\omega_S \otimes L) = H^0(\omega_S \otimes L)$.

But when $L$ is ”small” this is a subtle problem.

For example if $L = \mathcal{O}_S$ and $S$ is an elliptic curve, then $S^0(\omega_S) = H^0(\omega_S)$ iff and only if $S$ is ordinary.

Conjecturally, if $S$ is defined over $k$ of characteristic 0, then $S^0(\omega_{S_p}) = H^0(\omega_{S_p})$ for infinitely many primes.

Caution: this is not even known for abelian varieties.

Local versions of this conjecture are also interesting. Eg if $S$ is log canonical, then we expect that $S_p$ is **locally F-split** for infinitely many primes $p$ meaning that $F^e_* \omega_{S_p} \to \omega_{S_p}$ is surjective.
It is known that if $S$ is klt, then $S_p$ is **strongly F-regular** for all primes $p \gg 0$ (Hara, Mustata-Srinivas, Smith).

Here, strongly F-regular means that for any effective divisor $D \geq 0$, the induced map $F^e_* \omega_{S_p}(D) \to \omega_{S_p}$ is surjective for $e \gg 0$.

When $\dim S = 2$ and $p > 5$, Hara shows that klt singularities are exactly the strongly F-regular singularities.

We were able to leverage this result to prove the existence of flips for 3-folds.
Theorem (Hacon-Xu, Birkar, Waldron, Hacon-Witaszek)

Let $X$ be a smooth projective 3-fold over an algebraically closed field of characteristic $> 3$, then $R(\omega_X)$ is finitely generated and there exists a finite sequence of flips and divisorial contractions to a minimal model $X \to X_{\text{min}}$ (so that $X_{\text{min}}$ has terminal singularities and $\omega_{X_{\text{min}}}$ is nef).

- One of the key steps in the proof is to show the existence of pl-flips (Shokurov).
Recall that a pl-flip is a flipping contraction \( f : X \to \bar{X} \) with \( \rho(X/\bar{X}) = 1 \), \(-K_X - B\) and \(-S\) are ample over \( \bar{X} \) and \((X, S + B)\) is a plt pair.

In particular \( K_X \sim_{\mathbb{Q}} \lambda S \) for some \( \lambda > 0 \) (for simplicity we assume \( \bar{X} \) is affine and \( B = 0 \)).

To show the existence of the flip, we must show that \( R(K_X) \) is finitely generated (over \( \bar{X} \)).

Then the flip \( X^+ \to \bar{X} \) is given by \( X^+ = \text{Proj}_{\bar{X}}(R(K_X)) \).

This is equivalent to showing that \( R(K_X + S) \) is finitely generated.
From the short exact sequences

\[ 0 \to (\omega_X(S))^\otimes m(-S) \to (\omega_X(S))^\otimes m \to (\omega_S(B_S))^\otimes m \to 0 \]

where \( K_S + B_S = (K_X + S)|_S \), it follows that \( R(K_X(S)) \) is finitely generated if so is

\[ R_S(K_S + B_S) = \text{Im}(R(K_X + S) \to R(K_S + B_S)). \]

The rough idea is that the kernel of the above map is a principal ideal defined by the equations of \( S \).

Note that \((S, B_S)\) is a klt surface and so \( R(K_S + B_S) \) is finitely generated and hence the statement would follow if we can show that \( S^0(m(K_S + B_S)) = H^0(m(K_S + B_S)) \).

Loosely speaking, we achieve this by applying a generalization of Hara’s result.
Hara’s result applies for $p > 5$, however for $p = 5$ we have a detailed description and we can do a case by case analysis.

The cases $p = 2, 3$ or $d \geq 4$ seem extremely hard and I expect/hope that finite generation of the canonical ring will fail in higher dimensions and low characteristics.