Symmetries of polynomial equations

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- The automorphisms of the polynomial ring $\mathbb{C}[z]$.
- Replace $\mathbb{C}$ with the Riemann sphere $\mathbb{C} \cup \{\infty\}$.
- $z \longrightarrow \frac{az+b}{cz+d}$, $ad - bc \neq 0, \in \mathbb{C}$, the group of Möbius transformations.

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- $C = \mathbb{C}/\Lambda$ is a curve of genus 1, Lie group $S^1 \times S^1$.
- C acts on itself by translation, and $\mathrm{Aut}(C)$ is a finite extension of C. The dimension of $\mathrm{Aut}(C)$ is one.

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- The Wiman sextic C, given by

$$10x^3y^3 + 9(x^5 + y^5)z - 45x^2y^2z^2 - 135xyz^4 + 27z^6.$$

$\mathrm{Aut}(C) = A_6$. $|\mathrm{Aut}(C)| = 360$.

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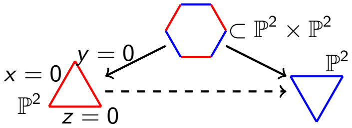

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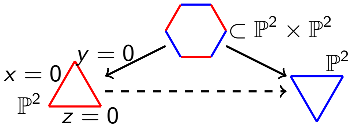

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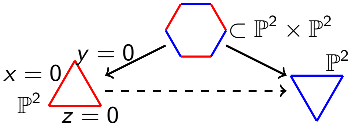

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- This Theorem is very deceptive.

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- Minimal rational surfaces S (Mori fibre spaces): $\mathbb{P}^2$, or a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$, $\mathbb{F}_n = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$.

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- Check: $\mathbb{F}_1 = \mathrm{Bl}_p \mathbb{P}^2$, $\dim \mathrm{Aut}(\mathbb{F}_1) = 8 - 2 = 6$.
- $\mathrm{Bir}(\mathbb{P}^2)$ is infinite dimensional; if we pick $f : \mathbb{P}^2 \dashrightarrow \mathbb{F}_n$, then $f^{-1} \mathrm{Aut}(\mathbb{F}_n) f \subset \mathrm{Bir}(\mathbb{P}^2)$.

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$$[x : y : z : t] \longrightarrow [x(t^d + f) : y(t^d + f) : z(t^d + f) : tf],$$

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- If the set R generates C_n then R must contain an element which blows down the cone over C.
- Any generating set is infinite dimensional, it must contain a copy of $\bigcup_g M_g$, $\dim M_g = 3g - 3$.

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- Proof: $\mathrm{Aut}(\mathbb{P}^n) = \mathrm{PGL}_{n+1}(\mathbb{C}) \subset \mathrm{GL}_N(\mathbb{C})$.
- Conjecture: (Serre) $\mathrm{Bir}(\mathbb{P}^n)$ is Jordan.

The MMP and the Jordan Property

- Theorem (Birkar; BCHM; Prokhorov and Shramov): C_n is Jordan.

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- $G \subset \mathrm{Aut}(X)$, $X \longrightarrow Z$, Z smaller dimension.

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- $G \subset \mathrm{Aut}(X) \subset \mathrm{Aut}(\mathbb{P}^N)$, which is Jordan.

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- Call the quotient $\operatorname{Aut}(X)/\operatorname{Aut}^0(X)$ the discrete part of the automorphism group (aka $\pi_0(\operatorname{Aut}(X))$).
- Theorem: Lesieutre There are examples of smooth projective varieties X whose discrete part is not finitely generated.

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$$K_C = \pi^*(K_B + \Delta),$$

where

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Case by case analysis. $(r_1, r_2, r_3) = (2, 3, 7)$ and $h = 0$ achieves bound $1/42$.

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- Question: Is the Wiman sextic the curve with the maximum number of automorphisms, amongst all smooth curves of genus 10?

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Review of finite simple groups

- Let $V = \mathbb{F}_{q^2}^m$. There is a sesquilinear pairing

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