Lecrute 1

SOLVABILITY OF EQUATIONS IN FINITE TERMS AND TOPOLOGICAL GALOIS THEOREM

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UNSOLVABILITY IN FINITE TERMS

What does it mean that an equation can not be solved explicitly? One can fix a class of functions and say that an equation is solved explicitly if its solution belongs to this class. Different classes of functions correspond to different notions of solvability.

A class of functions can be introduced by specifying:

a list of **basic functions** and

a list of **admissible operations**.

Given the two lists, the class of functions is defined as

the set of all functions that can be obtained from the basic functions by repeated application of admissible operations.

CLASSICAL CLASSES OF FUNCTIONS

To define a classical class of functions we have to fix its list of basic functions and its list of admissible operations.

Many of them use the **list of basic elementary functions** and the **list of classical operations**.

LIST OF BASIC ELEMENTARY FUNCTIONS

All constants, x (or $x_1, \dots, x_n$);

exp, ln, $x \to x^\alpha$;

sin, cos, tan;

arcsin, arccos, arctan.

LIST OF CLASSICAL OPERATIONS

1) Composition: $f, g \in L \Rightarrow f \circ g \in L$;

2) arithmetic operations: $f, g \in L \Rightarrow f \pm g, f \times g, f/g \in L$;

3) differentiation: $f \in L \Rightarrow f' \in L$;

4) integration: $f \in L$ and $y' = f$, i.e., $y = C + \int^{x} f(t)dt \Rightarrow y \in L$;

5) extension by exponent of integral: $f \in L$ and $y' = fy$, i.e., $y = C \exp \int^{x} f(t)dt \Rightarrow y \in L$;

6) algebraic extension: $f_1, \ldots, f_n \in L$ and $y^n + f_1 y^{n-1} + \cdots + f_n = 0 \Rightarrow y \in L$;

7) exponent: $f \in L$ and $y' = f'y$, i.e., $y = C \exp f \Rightarrow y \in L$;

8) logarithm: $f \in L$ and $dy = df/f$, i.e., $y = C + \ln f \Rightarrow y \in L$;

9) meromorphic operation: if $F : \mathbb{C}^n \to \mathbb{C}$ is a meromorphic function, $f_1 \ldots, f_n \in L$, and $y = F(f_1, \ldots, f_n) \Rightarrow y \in L$.

The operations 2) and 7) are meromorphic operations.

RADICALS, QUADRATURES, etc.

I. Radicals. Basic functions: rational functions.

Operations: arithmetic operations and extensions by radicals.

II. Elementary functions. Basic functions: basic elementary functions.
Operations: composition, arithmetic operations, differentiation.

III. Generalized elementary functions. The same as elementary functions + algebraic extensions.

IV. Quadratures. Basic functions: basic elementary functions.
Operations: composition, arithmetic operations, differentiation and integration.

IV′. "Liouville's quadratures". Basic functions: all complex constants. Operations: the arithmetic operations, integration, extension by the exponent of integral.

V. Generalized quadratures. The same as quadratures + algebraic extensions.

Liouville's Theorem. Class of "Liouville's quadratures" = class of quadratures.

Liouville's Theorem reduces the problem of solvability by quadratures to differential algebra: a function representable by quadratures can be constructed without use of highly non-algebraic operation of taking composition of two given functions.

The similar result holds for all classical classes of functions.

LIOUVILLE'S THEORY

Joseph Liouville (24.04.1809 – 8.09.1882, French mathematician)

The First Liouville Theorem (1833)

The theorem provides conditions for integrability of elementary functions in finite terms. For example, it shows that one can not write elementary formulas for the following integrals:

$$\int_a^x \frac{dx}{\sqrt{x(x+1)(x+2)}}$$

$$\int_a^x e^{-x^2} dx$$

$$\int_a^x \frac{dx}{\ln x}$$

The First Liouville Theorem for algebraic functions. *An integral $y(x)$ of an algebraic function is a generalized elementary function if and only if*

$$y(x) = A_0(x) + \sum_{i=1}^{n} \lambda_i \ln A_i(x),$$

where $\lambda_i \in \mathbb{C}$ and A_i are algebraic functions.

Similarly Liouville answered on the following question:

Which generalized elementary function has antiderivative representable by generalized elementary functions?

Slogan of Liouville's theory:

"Sufficiently simple" equations have either "sufficiently simple" explicit solutions or no explicit solutions at all.

The Seconds Liouville Theorem (1841) over the field of rational functions. *An equation*

$$y'' + py' + qy = 0,$$

where p, q are rational functions, is solvable by generalized quadratures if and only if it has a solution $y_1(x) = \exp \int^x a(t)dt$, where $a(t)$ is an algebraic function.

Example. The equation

$$y'' + xy = 0$$

is unsolvable by generalized quadratures.

In fact, Liouville proved more generale criterion of solvability by generalized quadratures of a second order linear differential equation over arbitrary differential field (not only over the field of rational functions).

Theorem (Picard–Vessio 1910, M. Rosenlicht 1973, Kh. 2018). *An equation* $L(y) = a_n y^{(n)} + \cdots + a_0 y = 0$, *where* a_i *belong to a differential field* K, *is solvable by generalized quadratures over* K *if and only if* $L(y)$ *is representable in the form*

$$L(y) = a_n \prod_{1 \le i \le n} L_i(y),$$

where $L_i(y) = y' - p_i y$ *and the element* p_i *is algebraic over* K.

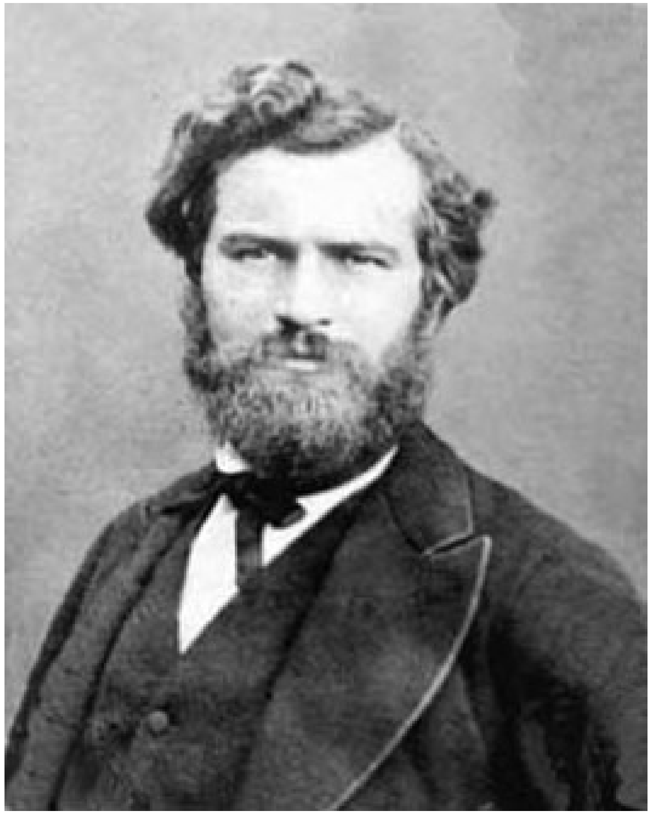
For $n = 2$, the above theorem is equivalent to the Second Liouville Theorem.

To prove the above Theorem Picard and Vessio developed the differential Galois theory. Rosenlicht used the valuation theory. My proof is based on the original ideas due to Liouville.

GALOIS THEORY

Évariste Galois (25.10.1811 – 31.04.1832, French mathematician)

Camille Jordan (5.01.1838 – 22.01.1922, French mathematician)



Galois discovered a criterion on solvability of an algebraic equation over a field K. He considered a field extension $K \subset F$ of K obtained by adjoining to K all roots of the algebraic equation. The **Galois group** G of the algebraic equation is the group of automorphisms of the pair (K, F) which fix the field K. Galois showed that **the equation is solvable by radicals over K if and only if its Galois group G is solvable.**

Actually Galois was killed when he was too young. He had no time to complete his theory. It was mainly developed by C. Jordan.

In particular, Jordan understood that the **Galois group of an algebraic equation over the field of rational** has topological meaning: it **is equal to the monodromy group of the algebraic function, defined by thee equation.**

PICARD–VESSIOT THEORY

Émile Picard (24.07.1856 – 11.12.1941, French mathematician)

Ernest Vessiot (8.03.1865 – 17.10.1952, French mathematician)

Picard discovered a similarity between linear differential equations and algebraic equations. He considered a differential field extension $K \subset F$ of K obtained by adjoining to K all solutions of the linear differential equation. The **Galois group** G of the linear differential equation is the group of automorphisms of the pair (K, F) which fix K.

Theorem (Picard–Vessiot, 1910). *A linear differential equation over a differential field K is solvable by quadratures if and only if its Galois group is solvable. It is solvable by generalized quadratures if and only if the connected component of the identity in its Galois group is solvable.*

Picard–Vessiot theory has many applications. For example, for an equation whose coefficients are rational functions with integral coefficients, it allows to determine explicitly is the equation solvable by generalized quadratures or not.

TOPOLOGICAL GALOIS THEORY

Theorem (C. Jordan). The Galois group of an algebraic equation over the field of rational functions in several complex variables is isomorphic to the monodromy group of the (multivalued) algebraic function defined by the same equation.

Jordan's theorem implies that the Galois group of an algebraic equation over the field of rational functions in several complex variables has a pure topological meaning.

One-dimensional topological Galois theory deals with functions in one variable.

There is also a multidimensional version of topological Galois theory, but we will not talk about it now.

CONSTRUCTING TOPOLOGICAL

GALOIS THEORY

Program:

I. Find a wide class of functions which is closed under classical operations, such that for all functions from the class the monodromy group is well defined.

II. Use the monodromy group within this class instead of the Galois group.

CLASS OF $\mathcal{S}$-FUNCTIONS

A multivalued analytic function of one complex variable is called $\mathcal{S}$-function if the set of its singular points is at most countable.

Theorem. *The class of $\mathcal{S}$-functions is closed under:*

1) composition;

2) arithmetic operations;

3) differentiation;

4) integration;

5) meromorphic operations;

6) solving algebraic equations;

7) solving linear differential equations.

Corollary. *A function, representable by generalized quadratures, is $\mathcal{S}$-function.*

Thus, a function having an uncountable number of singular points can not be expressed by generalized quadratures.

Example. Consider a function

$$f = \ln(\sum_{i=1}^{n} \lambda_i \ln(x - a_i)).$$

If $n \geq 3$, λ_i are generic, and $a_i \neq a_j$ if $i \neq j$, then:

1) the monodromy group of f contains continuum elements;

2) the set of singular points of f is everywhere dense on the complex line.

SOLVABLE MONODROMY GROUP

Theorem. *The class of $\mathcal{S}$-functions, whose monodromy group is solvable, is closed under:*

1) integration;

2) differentiation;

3) composition;

4) meromorphic operations (in particular, arithmetic operations).

Theorem. *If the monodromy group of a function f is unsolvable, then f cannot be represented via meromorphic functions using integration, differentiation, composition and meromorphic operations.*

Corollary. *If the monodromy group of an algebraic equation whose coefficients are rational functions is unsolvable, then its solutions cannot be represented via meromorphic functions using integration, differentiation, composition, and meromorphic operations.*

Corollary. *If the monodromy group of a linear differential equation (or a system of linear differential equations), whose coefficients are rational functions, is unsolvable, then its solutions cannot be represented via meromorphic functions using integration, differentiation, composition and meromorphic operations.*

SOLUTIONS OF ALGEBRAIC EQUATIONS AND OF FUCHS-TYPE DIFFERENTIAL EQUATIONS

A linear differential equation (or a system of linear differential equations), whose coefficients are rational functions, is of **Fuchs-type** if its solutions have a polynomial growth when the argument approaches poles of the coefficients along a smooth curve.

Theorem. *If the monodromy group of an algebraic function is solvable, then by Galois theory it can be represented by radicals.*

If the monodromy group of a Fuchs-type linear differential equation (or of a system of Fuchs-type linear differential equations) is solvable, then this equation or this system of equations is solvable by quadratures.

INTEGRABILITY CONDITIONS FOR FUCHS-TYPE SYSTEMS WITH SMALL MATRICIDES A_i

Theorem. *Consider a system*

$$y' = \sum \frac{A_i}{x - a_i} y,$$

where y is n-vector, and A_i are $n \times n$ matrices with constant entries.

Assume that the matrices A_i have sufficiently small entries. Then the system can be solved by quadratures if and only if all the matrices A_i are triangular in some basis.

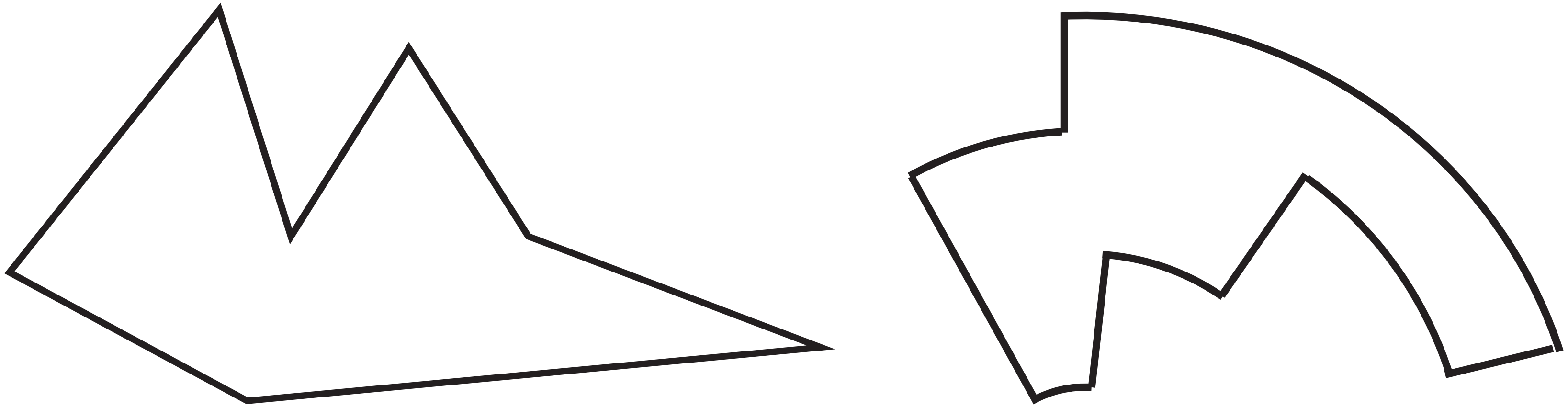
Moreover, if such system is not triangular in some basic, one can not write an finite formula for its generic solution which uses arbitrary meromorphic functions, compositions, integrations and solutions of algebraic equations.

A MAP FROM A BALL TO A POLYGON WHOSE SIDES ARE ARCS OF CIRCLES AND SEGMENTS

Let G be a polygon on the Riemann sphere bounded by arcs of circles and by segments. Let $f_G : B_1 \to G$ be a Riemann map from a unit ball onto G. One can classify all polygons G such that the function f_G is representable by generalized quadratures.

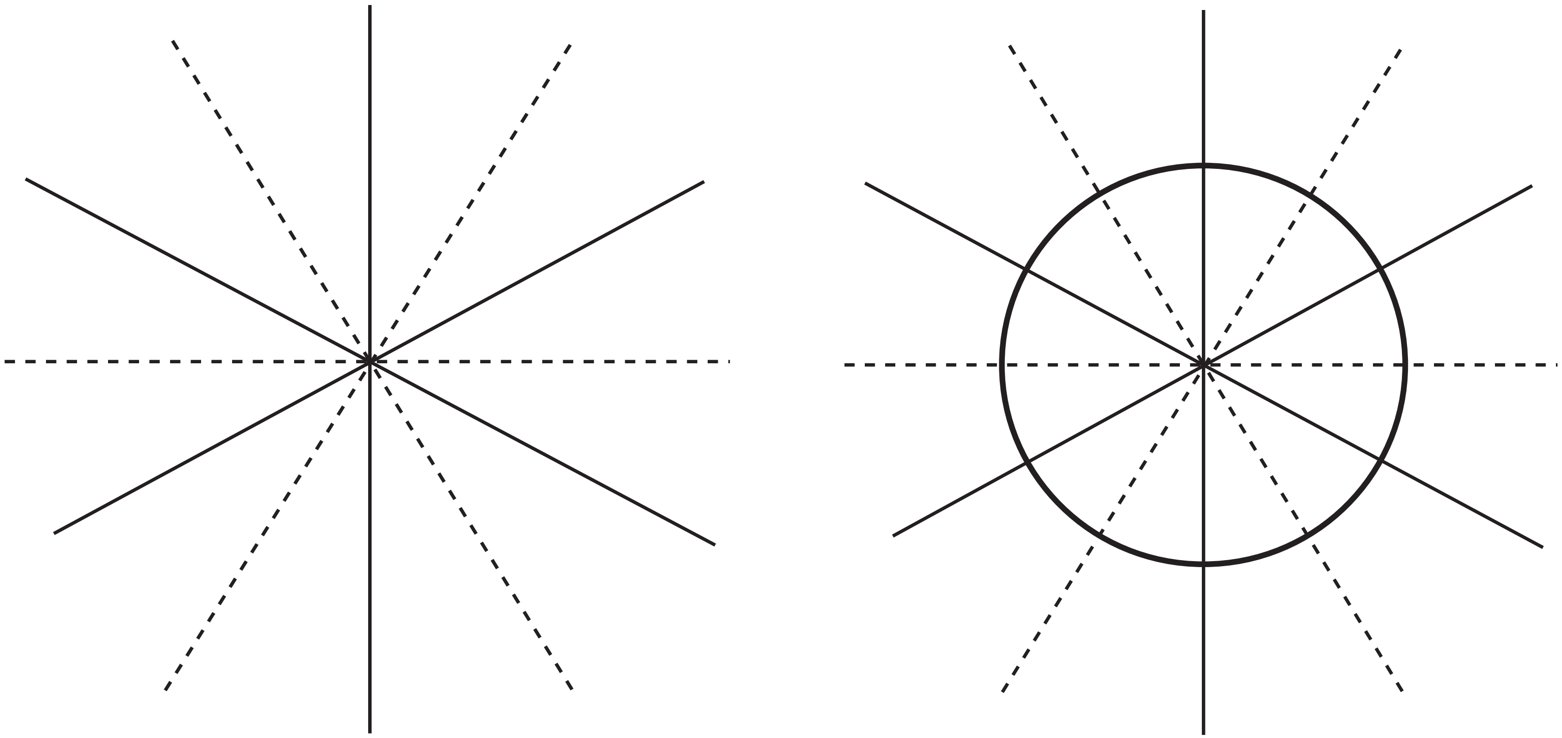
Below we present such classification up to a fraction-linear transformation of the complex line.

If a polyhedron G is not listed in the classification, then the function f_G cannot be represented by a finite formula which uses arbitrary meromorphic functions, compositions, integrations and solutions of algebraic equations.



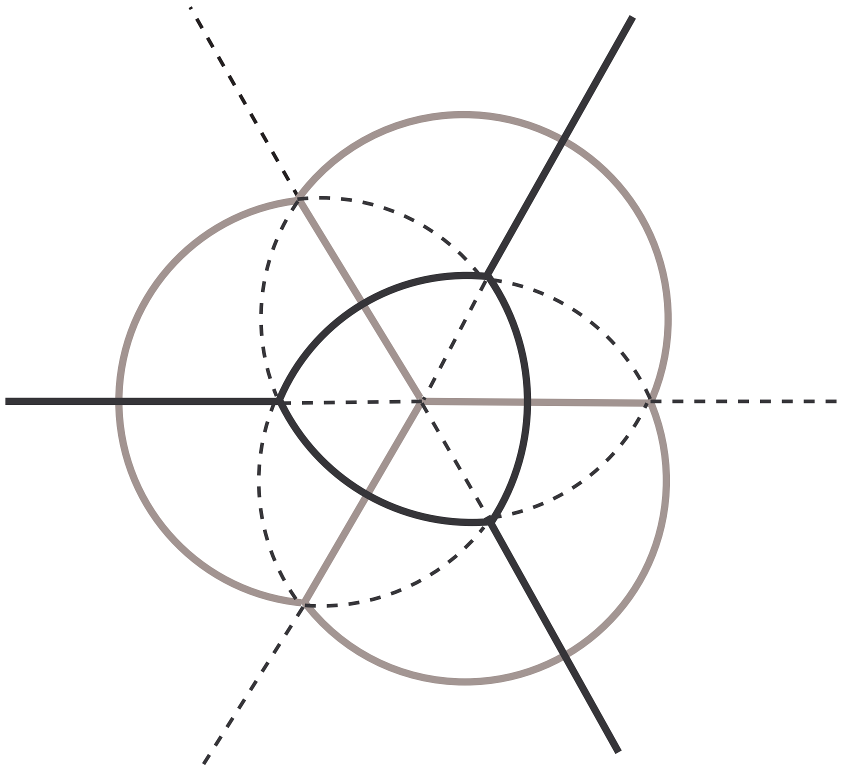
On the left: First case of integrability: all sides of G pass through one point (on the diagram this point is ∞.

On the right: Second case of integrability: there are two points such that a side of G either passes through the points, or the points are symmetric about the side (on the diagram this points are ∞ and zero).

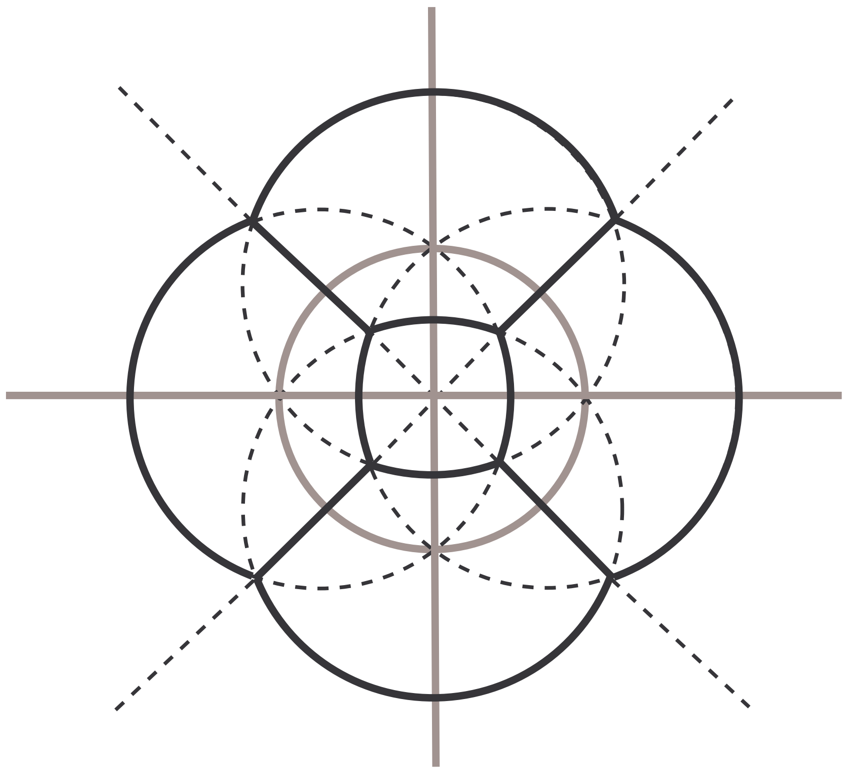


THIRD CASE OF INTEGRABILITY

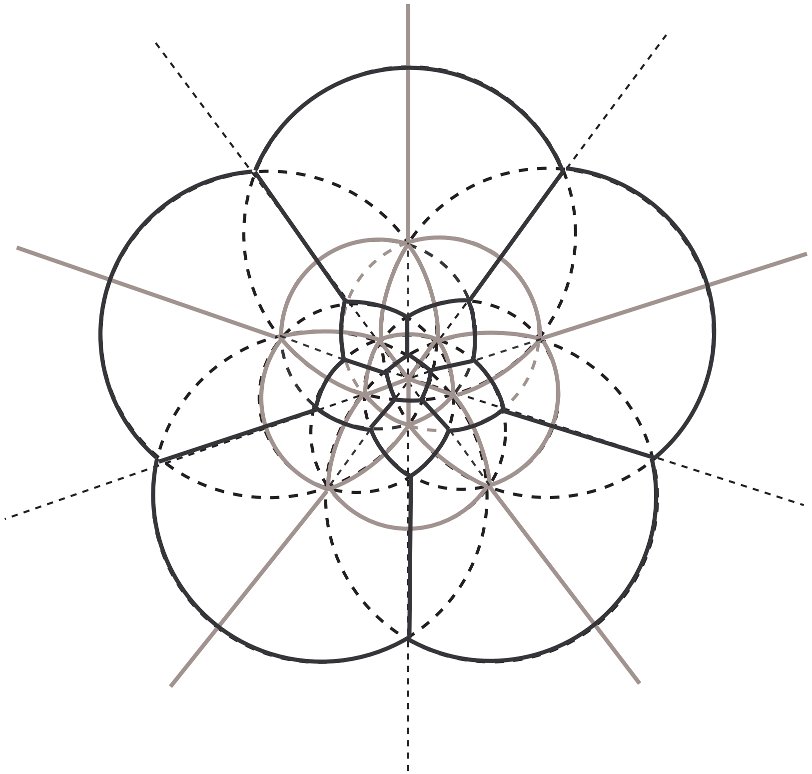
The sides of the polygon G are located on finite nets of circles obtained by the stereographic projection of a finite net of great circles, such that the group generated by reflections in these circles is finite.



Symmetries of regular tetrahedron



Symmetries of regular cube regular octahedron



Symmetries of regular dodecahedron – regular icosahedron

POLYNOMIALS INVERTIBLE IN k-RADICALS

Theorem (J.F. Ritt 1922). *A polynomial is invertible in radicals if and only if it is a composition of the power polynomials* $z \to z^n$, *Chebyshev polynomials, and polynomials of degree* ≤ 4.

Theorem (Yu. Burda, Kh. 2012). *A polynomial is invertible in* k*-radicals, i.e, is invertible in radicals and solutions of equations of degree at most* k, *if and only if is a composition of power polynomials, Chebyshev polynomials, polynomials of degree at most* k *and if* $k \leq 14$, *certain exceptional polynomials (a complete list of such polynomials is known).*

The proof is based on classification of finite simple groups and results on primitive polynomials obtained by Muller and many other authors.

13-th HILBERT'S PROBLEM

The general degree n algebraic function $x(a_0, \dots, a_{n-1})$ is the solution of the equation $x^n + a_{n-1}x^{n-1} + \dots + a_0 = 0$.

Problem (D. Hilbert). *Find the smallest $H(n)$, such that x can be represented by composition of algebraic functions of $H(n)$ variables.*

More generally: *Which algebraic functions of n variables can be represented by composition of algebraic functions of $m < n$ variables?*

Actually, the problem on compositions was formulated by Hilbert for continuous functions, not for algebraic functions.

Theorem (Kolmogorov, Arnold 1957). *Any continuous function of n variables can be represented as the composition of functions of a single variable with the help of addition.*

ALGEBRAIC FUNCTIONS OF ONE VARIABLE

In Kolmogorov–Arnold Theorem, one cannot replace continuous functions by entire algebraic functions.

Theorem (Kh. 1969). *If an algebraic function can be represented as a composition of polynomials and entire algebraic functions of one variable, then its local monodromy group at each point is solvable.*

Sketch of the proof. At each point, the local monodromy group of an algebraic function of one variable is a cyclic group. The operation of division that destroys locality is not an allowed operation in Theorem. Now Theorem follows from the Galois theory type arguments.

COROLLARY AND OPEN PROBLEM

Corollary. *The function* $y(a, b)$, *defined by equation*

$$y^5 + ay + b = 0,$$

cannot be expressed in terms of entire algebraic functions of a single variable by means of composition, addition and multiplication, since its local monodromy group at the origin is the unsolvable group $S(5)$ *of all permutations of five elements.*

It is easy to see that $y(a, b) = g(b/\sqrt[4]{a^5})\sqrt[4]{a}$, where $g(u)$ is defined by equation $g^5 + g + u = 0$.

Problem (still open!) *Show that there is an algebraic function of two variables which cannot be expressed in terms of algebraic functions of a single variable by means of composition and arithmetic operations.*

LITERATURE

Materials of the lecture related to the Liouville's theory can be found in the following book and papers:

[1] Ritt, J.F. *Integration in Finite Terms. Liouville's Theory of Elementary Methods.* Columbia University Press, New York, NY. 1948.

[2] Rosenlicht, M. *An analogue of l'Hospital's rule.* Proc. Amer. Math. Soc. 37:369–373. 1973.

[3] Khovanskii, A. *On Algebraic Functions Integrable in Finite Terms.* Funct. Anal. Appl. V. 49, No 1, 62–70, 2015.

[4] Khovanskii, A. *Comments on JF Ritt's Book "Integration in Finite Terms".* in book: Integration in Finite Terms: Fundamental Sources by Clemens Raab (Editor), Michael F. Singer (Editor). Springer. Series: Text & Monographs in Symbolic Computation. 2022. 137–201.

[5] Khovanskii, A. and Tronsgard, A. *On solvability of linear differential equations in finite terms.* Uspekhi Mat. Nauk; translation in Russian Math. Surveys 14 pp. (accepted)

The differential Galois theory is presented in:

[6] Marius van der Put and Michael Singer. *Galois theory of linear differential equations, volume 328 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences].* Springer-Verlag, Berlin. 2003.

Materials of the lecture related to the topological Galois Theory can be found in:

[7] Khovanskii, A. *Topological Galois Theory.* Springer-Verlag, Heidelberg. 2014.

[8] Burda, Yu. & Khovanskii, A. Polynomials invertible in k-radicals. Arnold Mathematical Journal, V. 2, No 1, 2016, 121–138.