GROUPS OF SMALL TYPICAL DIFFERENTIAL DIMENSION

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Abstract. We apply techniques from ω-stable and superstable groups to strongly connected and almost simple differential algebraic groups in the sense of Cassidy and Singer. We analyze differential algebraic group actions from this point of view, and prove several results regarding interpreting fields from these actions. We prove a differential algebraic analogue of Rienecke's theorem. We show that every strongly connected differential algebraic group with typical differential dimension two is solvable. A special instance of the Cassidy-Singer conjecture is confirmed. Namely, noncommutative almost simple groups of typical differential dimension 3 are equal to $SL_2(F)$ or $PSL_2(F)$ for a definable subfield F.

1. Introduction

This paper aims to apply techniques from ω-stable groups and groups of finite Morley rank to prove results about differential algebraic groups. Of course, this seems like a strange goal since differential algebraic groups *are* ω-stable and many are of finite Morley rank. But, our results are not stated model theoretically, nor do they have model theoretic hypotheses. We do not use Morley rank or U-rank (or any onther model theoretic notions) in the statements of the results, nor are there known lower bounds for these ranks with respect to the notions of dimension we use. For some discussion of this issue, see [19] [7].

Zilber's indecomposability theorem is a powerful tool for proving definability results in groups of finite Morley rank. Generalizations of Zilber's theorem to the superstable case exist [5], even in the infinite rank case [2]. A related theorem is given in [7] for differential algebraic groups. We will show how to prove similar definability results in the differential algebraic setting using the indecomposability theorem. The theorem is one of the key tools for carrying out a detailed analysis of groups of small Morley rank. We aim for an analysis of differential algebraic groups of small typical differential dimension. Specifically, our analysis is similar in spirit to portions of chapter seven of [10] and [4] with Morley rank replaced by typical differential dimension. Of course, complications arise since the finiteness conditions in our setting are not nearly as strong as those when dealing with finite Morley rank objects.

In section two, we consider some general interpretability and definability results in differential algebraic groups. Our analysis concentrates on the question of interpreting definable fields via given differential algebraic groups or differential algebraic group actions. Section three begins the analysis of differential algebraic groups with the view that Morley rank is to connected groups of finite Morley rank as typical differential dimension is to strongly connected differential algebraic groups. Groups of typical differential dimension one or two are considered in section three. In section four, we add some group theoretic assumptions and obtain stronger results. The final section consists of remarks on an open problem and points out how the author's earlier work [7] can be used to give an answer in a special case.

The notation of this paper comes from general model theoretic and differential algebraic conventions, but some of the notation was recently invented in [3] and [7]. Both model theory and differential algebra use the word *type*. The meanings are different, but the meanings are different enough so that the distinction is usually clear. In differential algebra, *type* refers to the degree of the Kolchin polynomial of a tuple over a specified base differential field. In this paper, we will always assume we are working over some fixed differential field over which the groups and varieties in question are defined. In the model theory of differential fields, the *type* of a tuple is the collection of all first order statements true about that tuple over a given differential field.

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2. Definitions and notation

The model theory of partial differential fields of characteristic zero with finitely many commuting derivations was developed in [13]. We only consider commuting derivations. We will always denote by Δ, a finite set of distinguished derivations. Characteristic zero differential fields have a model companion, which we denote $DCF_{0,m}$. For a more recent alternate (geometric) axiomatization of partial differentially closed fields, see [18]. For a reference in differential algebra, we suggest [8] and [11]. The theory $DCF_{0,m}$ has *quantifier elimination*, so the definable groups are *differential algebraic groups*. By a result of Pillay [15], the differential algebraic groups are precisely the definable groups in this setting. Though Pillay's proof occurs in the ordinary differential setting, all of the arguments work in the partial differential setting as well.

$DCF_{0,m}$ has *elimination of imaginaries*, so quotients of differential algebraic groups are again differential algebraic groups. Throughout this paper, $K \models DCF_{0,m}$ will be a field over which our sets are defined and $\mathcal{U}$ is a very saturated model of $DCF_{0,m}$. All tuples in differential field extensions of K can be assumed to come from $\mathcal{U}$. $\mathcal{U}$ functions as a universal differential field in the sense of Kolchin.

Quantifier elimination also gives a bijective correspondence between varieties, types, and radical differential ideals. Given a type $p \in S(K)$, we have a corresponding differential radical ideal via

$$p \mapsto I_p = \{f \in K\{y\} \mid f(y) = 0 \in p\}$$

Of course, the corresponding variety is simply the zero set of I_p. We will use this correspondence implicitly throughout including in the notation for Kolchin polynomials, which we will define next.

Let Θ be the free commutative monoid generated by Δ. For $\theta \in \Theta$, if $\theta = \delta_1^{\alpha_1} \dots \delta_m^{\alpha_m}$, then $ord(\theta) = \alpha_1 + \dots + \dots + \alpha_m$. The order gives a grading on the monoid. $\Theta(s) = \{\theta \in \Theta : ord(\theta) \leq s\}$.

Kolchin proved the following, [8] Theorem 6, page 115.

Theorem 2.1. *Let $\eta = (\eta_1, \dots \eta_n)$ be a finite family of elements in some extension of k. There is a numerical polynomial $\omega_{\eta/k}(s)$ with the following properties.*

1. *For sufficiently large $s \in \mathbb{N}$, the transcendence degree of $k((\theta\eta_j)_{\theta \in \Theta(s), 1 \leq j \leq n})$ over k is equal to $\omega_{\eta/k}(s)$.*
2. *$deg(\omega_{\eta/k}(s)) \leq m$*
3. *One can write*

$$\omega_{\eta/k}(s) = \sum_{o \leq i \leq m} a_i \binom{s+i}{i}$$

   In this case, a_m is the differential transcendence degree of $k\langle\eta\rangle$ over k.

4. *If $\mathfrak{p}$ is the defining differential ideal of η in $k\{y_1, \dots, y_n\}$ and Λ is a characteristic set of $\mathfrak{p}$ with respect to an orderly ranking of $(y_1, \dots, y_n)$, and if for each y_j we let E_j denote the set of all points $(e_1, \dots, e_m) \in \mathbb{N}^m$ such that $\delta_1^{e_1} \dots \delta_m^{e_m} y_j$ is a leader of an element of Λ, then*

$$\omega_{\eta/k}(s) = \sum_{1 \leq j \leq n} \omega_{E_j} - b$$

   where $b \in \mathbb{N}$.

The Kolchin polynomial of a differential variety is not a Δ-birational invariant, but the leading coefficient and the degree are Δ-birational invariants. We call the degree the *differential type* or Δ-type of V. We will use the notation $\tau(V)$ for the the differential type. Noting the above correspondence between tuples in field extensions (realizations of types) and varieties, we will occasionally write $\tau(p)$ or $\tau(a)$. The leading coefficient is called the *typical differential dimension* or the typical Δ-dimension. We will write $a_\tau(V)$ for the typical differential dimension of V. We will also write $a_\tau(a)$ and $a_\tau(a)$ for a tuple of elements a in a field extension. Similarly, we write $a_\tau(p)$ and $a_\tau(p)$ for a complete type p. For further results on the signifigance of Kolchin polynomials, see [8] and [9].

In what follows, we will pay little attention to differential type. This approach is in contrast to the results of [3] in which work is done under the assumption of differential type one. Instead, we will restrict the typical differential dimension, but allow arbitrary differential type.

The following is elementary to prove, see [14].

Lemma 2.2. *For a, b in a field extension of K.*

$$\tau(a,b) = max\{\tau(a), \tau(tp(a/b))\}$$

If $\tau(a) = \tau(tp(b/a))$, then

$$a_\tau((a,b)) = a_\tau(a) + a_\tau(b/\{a\} \cup K)$$

If $\tau(a) > \tau(tp(b/a))$, then

$$a_\tau((a,b)) = a_\tau(a)$$

The following definition (due to Cassidy and Singer) gives us the basic objects of study in this paper.

Definition 2.3. A differential algebraic group G is *strongly connected* if $\tau(G/H) = \tau(G)$ for every differential algebraic subgroup, H. G is almost simple if $\tau(H) < \tau(G)$ for every differential algebraic subgroup, H.

Strongly connected and almost simple differential algebraic groups abound in this setting; for instance, in order for G to have regular generic type, it is clearly necessary that G be almost simple. However, the precise relationship between regularity and almost simplicity is not entirely clear (see the questions raised in [14], for instance). Every differential algebraic group G has a characteristic subgroup which is the largest strongly connected differential algebraic group, called the *strongly connected component*. Any strongly connected group has a series of subnormal differential algebraic groups such that the successive quotients are almost simple.

Example 2.4. If H is a quasi-simple algebraic group and C' is a definable subfield, then $H(C')$ is an almost simple differential algebraic group. For a proof, see [3].

Example 2.5. The counterexamples of Suer [19] are almost simple. For instance, the zero locus of

$$\delta_1 x - \delta_2^2 x$$

In general, for a discussion almost simple groups and linear differential operators, see [7].

Example 2.6. The following example is due to Cassidy and Singer. Consider the following matrix group, G_n:

$$\begin{pmatrix} a & 0 & 0 \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$$

where $a \neq 0$, and $a^{-1}\delta a = \delta^n(b)$. The example is especially interesting since the groups G_n are all nonisomorphic, but they are isogenous. The following is an isogeny from $G_n \to \mathbb{G}_a$:

$$\begin{pmatrix} a & 0 & 0 \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \mapsto a^{-1}\delta(a)$$

Example 2.7. The following matrix group is not almost simple (or strongly connected):

$$\begin{pmatrix} 1 & u_1 & u \\ 0 & 1 & u_2 \\ 0 & 0 & 1 \end{pmatrix}$$

where $\delta_i(u_i) = 0$. The strongly connected component is the subgroup of matrices of the form:

$$\begin{pmatrix} 1 & 0 & u \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Though this paper does not use model theoretic notions of rank, we mention the following because much of this paper is inspired by techniques in groups of finite Morley rank. The following result is due to McGrail [13],

Theorem 2.8. *For any complete type in $DCF_{0,m}$,*

$$RU(p) \leq RM(p) \leq \omega^{\tau(p)}(a_\tau(p)+1)$$

The first inequality is true in any stable theory. The second inequality can be proved using Kolchin's theory (chapter 2 of [8]). There are other notions of dimension in differentially closed fields (specifically, one analagous to Krull dimension) which are explored and bounded in [17]. There is no known lower bound for Lascar rank in terms of differential type and typical differential dimension. Because of this, we can not apply many portions of the rich theory of superstable groups (for instance see [16]) without making assumptions about the model theoretic ranks.

3. Differential Algebraic Group Actions

In this section, we discuss differential algebraic groups in the sense of Kolchin [8]. Pillay showed that these are the definable groups in the theory $DCF_{0,m}$, see [15]. So, in this section, T will be $DCF_{0,m}$. Sometimes we will refer to the definable sets in this setting as *constructible sets of the Kolchin topology* or simply as constructible sets. Since there is no known lower bound for Lascar rank in terms of differential algebraic data (see [19]), there is not currently a way to apply the α-indecomposability theorem [2] to differential algebraic groups unless model theoretic hypotheses are assumed (or very special cases are considered). However, [7] gave an indecomposability theorem in the differential algebraic setting, with purely differential algebraic hypotheses and conclusions. The theorems of this section are differential algebraic analogues of results originally proved by Zilber [20] and generalized by [2]. We suggest Marker's exposition of these results, see chapter seven of [10]. For many examples of the groups discussed in this section, see [3]. The notation of this section follows [7] and [3]. The reader should note that in this section the definitions of n-connected, n-indecomposable, etc. are differential algebraic in nature, and might not have anything to do with model theory. For further discussion of this issue, see [7]. Since it is the degree of the Kolchin polynomial, n may be assumed to be a natural number, see [8] and [3].

Lemma 3.1. *Suppose that an strongly connected differential algebraic group Γ acts definably and transitively on a constructible set of Δ-type less than $\tau(\Gamma)$. Then the set is a singleton.*

Proof. The kernel of the action, H, must be of Δ-type and typical Δ-dimension equal to that of Γ. This is impossible since it forces $\tau(\Gamma/H) < \tau(\Gamma)$. □

Lemma 3.2. *Let G be a strongly connected differential algebraic group and $\sigma : G \to G$ a definable group homomorphism such that $(Ker(\sigma))$ has Δ-type less than $\tau(G)$. Then σ is surjective.*

Proof. Since $\tau(ker(\sigma)) < \tau(G)$, $\tau(\sigma(G)) = \tau(G)$ and $a_\tau(\sigma(G)) = a_\tau(G)$. This is impossible unless $\sigma(G) = G$. □

Theorem 3.3. *$(H, \cdot)$ and $(A, +)$ are infinite abelian differential algebraic groups such that H acts definably and faithfully on A where H acts as a group of automorphisms. Assume A and H are both Δ-type n. Assume that no subgroup $B \leq A$ of Δ-type at least n is H-invariant. Then $(H, \cdot)$ and $(A, +)$ interpret an algebraically closed field of the same Δ-type as H.*

Proof. The structure of the proof is much like that of Theorem 7.3.9 of [10]. Several portions of the proof given there apply directly here, but are reproduced for convenience. Many of the dimension theoretic arguments must be converted to the differential algebraic setting. First, we know that A is strongly connected, since the strongly connected component is automorphism invariant. Now, without loss of generality, we may let $a \in A$ be generic (this would only require connectedness, not strong connectedness).

Claim 3.1. $\tau(Ha) \geq n$

Proof. Let $H^{(0)}$ be the strongly connected component. Suppose $H^{(0)}a$ is of type less than n. Then by 3.1 $H^{(0)}a = \{a\}$. But, since A is connected,

$$X = \{x \in A \mid H^{(0)}x = \{x\}\}$$

is generic. Any element in A is a product of generics, so $H^{(0)}$ must fix all $c \in A$. The fact that H acts faithfully means that $H^{(0)} = \{1\}$. But, this means that $\tau(H) < n$, contradicting the hypotheses of the theorem. □

Claim 3.2. $Ha \cup \{0\}$ is n-indecomposable.

Proof. 3.2 $Ha \cup \{0\}$ is H-invariant, so we need only test the indecomposability for H-invariant subgroups (see [7]). But, by the hypotheses of the theorem, each of these have type less than n, so the conclusion follows. □

By the indecomposability theorem (see [7]) the subgroup $\langle Ha \cup \{0\}\rangle$ is definable and H-invariant. But then $\langle Ha \cup \{0\}\rangle = A$. Further, the proof of the theorem tells us that any $a \in A$ is the sum of k elements from Ha, for some fixed k. Consider the ring of endomorphisms of A. Since $H \subseteq End(A)$, we may define R to be the subring generated by H. Since H is abelian, R is commutative. For all $b \in A$, we know,

$$b = \sum_{i=1}^{m} h_i a$$

for some $h_i \in H$ and $m \leq k$. Now,

$$r(b) = \sum_{i=1}^{m} r(h_i a) = \sum_{i=1}^{m} h_i(ra)$$

so if $r_1 \neq r_2 \in R$, then it must be the case that $r_1 a = r_2 a$. Further, since if $ra = b$, then as above there are $h_i \in H$, such that

$$ra = b = \sum_{i=1}^{m} h_i a,$$

and so we can see

$$r = \sum_{i=1}^{m} h_i.$$

That is, every element of R is the sum of less than or equal to k elements of $H \cup \{0\}$. Using this, we can show that R is interpretable. Consider, on $(H \cup \{0\})^n$ the equivalence relation defined by $(h_1, \ldots, h_k) \sim (g_1, \ldots, g_k)$ if and only if $\sum h_a = \sum g_i a$. Naturally, we define $\bar{h} \oplus \bar{g} = \bar{l}$ if and only if $\sum h_i a + \sum g_i a = \sum l_i a$. Also, we define $\bar{h} \otimes \bar{g} = \bar{l}$ if and only if $\sum \sum h_i g_i a = \sum l_i a$. Then $R \cong ((H \cup \{0\})^n / \sim, \oplus, \otimes)$.

Claim 3.3. R is a field.

Proof. 3.3 Given any $r \in R$, with $r \neq 0$, take $b\ inA$ with $rb = 0$. Then $\forall h \in H$, $r(hb) = (rh)(b) = (hr)(b) = h(rb) = 0$. That is, $Ker(r)$ is H-invariant. But, we know that $\tau(Ker(r)) < n$. Now we can apply Lemma 3.2. So, there is some $c \in A$ with $rc = a$. Then for some $h_i \in H$,

$$c = \sum h_i a$$

so

$$r \sum h_i a = a.$$

But, as we have seen, an element of the ring R is uniquely determined by its action on a. But, then $r \sum h_i = 1$. □

By the superstable analogue of Macintyre's theorem, a superstable field is algebraically closed (see [5] or see [10] for the ω-stable version). In the differential setting, we know precisely that any interpretable field is actually the kernel of some set of C-linear combinations of the derivations of Δ, where C is the field of absolute constants [19]. Thus, in this case, the group A is isomorphic to the additive group of such a definable field. □

Remark 3.4. Moshe Kamensky notices that by the above construction, H is forced to be a subgroup of the multiplicative group of the field which we interpret. So, the interpretable fields are all kernels of some subset of linear combinations (over the absolute constants) of the distinguished derivations. The additive and multiplicative groups of these fields are almost simple. So, it is impossible for H to be a proper subgroup of the multiplicative group, thus, the only way to satisfy the hypotheses of the theorem is for H to be isomorphic to the multiplicative group of the field.

The proof of the following theorem almost identical to the proof given in [10] 7.3.12, with the appropriate changes ("finite" translates as "of lower Δ-type").

Theorem 3.5. *If G is a strongly connected solvable Δ-group with $Z(G)$ of Δ-type strictly less than G, then G interprets an algebraically closed field of the same Δ-type as G.*

Proof. We will do induction on the typical differential dimension of G. Since the type of $Z(G)$ is strictly less than that of G, we know that $G/Z(G)$ is a strongly connected (Cassidy and Singer prove that quotients of strongly connected groups are strongly connected [3]), centerless differential algebraic group of the same type and typical dimension as G. To see that $G/Z(G)$ is centerless: let $a \in G$ be such that $aZ(G)$ is in $Z(G/Z(G))$. Then for any $g \in G$, $a^{-1}g^{-1}ag \in Z(G)$, since a is central in the quotient. This means that $a^{-1}a^G$ is in the center of G. But, a^G is then of lower differential type, via the hypotheses of the theorem. But, we have a definable bijection $a^G \to G/C_G(a)$. Since G is strongly connected, $C/C_G(a)$ being of lower differential type means that it must be the identity. So, without loss of generality, we will assume that G itself is centerless (the quotient is clearly interpretable).

Now, we take A to be a minimal definable normal subgroup which is of the same differential type as G. By the minimality condition, A is almost simple. In [7], we showed that the commutator of a strongly connected group is a differential algebraic subgroup of the same type or the group is perfect. But, A is solvable and almost simple, so we must have that $[A, A] = 1$, that is A is commutative.

Next, we consider $C_G(A)$, which must not be all of G, since G is centerless and A is commutative. So, we let $G_1 = G/C_G(A)$. Then G_1 inherits the G action on A by conjugation, since under this action $C_G(A)$ is the kernel. Since we quotiented by

$C_G(A)$, the action is faithful. By the minimality conditions on A, we know that there are no invariant subgroups of A with the same differential type. Of course, G_1 is also solvable and of typical differential dimension less than that of G, since $A \subseteq C_G(A)$ is of the same differential type as G (for behaviour of typical differential dimension in quotients, see [14] or [3]).

Now, if G_1 has a center of strictly lower differential type, we may apply induction and interpret a field. So, assume $\tau(Z(G_1)) = \tau(G_1)$. Thus, we will let H be a minimal definable subgroup of $Z(G_1)$ of the same differential type. Then, again, H acts faithfully on A via conjugation. In the case that there are no proper H-invariant differential algebraic subgroups of A,, we may apply Theorem 3.3 to get the conclusion. If not, then we let B be a proper definable H-invariant subgroup of A. Then let H_0 be the subgroup of H which acts trivially on B. If $H = H_0$, then, because we know B is a minimal H-invariant subgroup, we know B is indecomposable (in the sense of [7]) and so are the groups B^g for all $g \in G_1$. But, since $H \subseteq Z(G_1)$, we know that for all $g \in G_1$, $h \in H$, and $b \in B$, $b^g = (b^g)^h$.

So, we can see H acts trivially on B^g. Then by the indecomposability theorem for differential algebraic groups, $\langle B^g \mid g \in G_1 \rangle$ is a definable G_1-invariant subgroup of A of the same differential type (since B is of the same type as A). This is impossible unless the group generated by the conjugates of B is actually A. But, H acts trivially on each conjugate of B and H does not permute the conjugates. So, H acts trivially on A.

So, now we know that H_0 is a proper subgroup of H. But, by the minimality assumptions on H, we know H_0 must be of lower type. So, taking the quotient H/H_0, we have a faithful H/H_0 action on B with no invariant subgroups of the same type as B. Note that since we are taking the quotient by a subgroup of lower type (H_0), if there were no invariant subgroups of differential type equal to B before taking the quotient, then there are none after taking the quotient. Now we apply Theorem 3.3 to interpret an algebraically closed field. □

The next two lemmas are due to Berline [1] in the superstable context and previously appeared at least in the work of Zilber and Cherlin in the finite Morley rank context. The partial differential version of the first lemma appears in [3] and [7]. The groups which appear in are assumed to be differential algebraic groups.

Lemma 3.6. *If H is a strongly connected subgroup of G is a strongly connected Δ-group, then for all $g \in G$, the Δ-type of g^H is either equal to the Δ-type of H or $H \leq C_G(g)$.*

Proof. There is a bijection between g^H and $H/C_G(g) \cap H$. H is strongly connected, so quotients of lower type are trivial. □

A natural consequence of this lemma is that every normal Δ-subgroup of smaller Δ-type in a strongly connected Δ-group is actually central, which was noted in [3].

Definition 3.7. A group is nilpotent if there is a chain of normal subgroups

$$G = G_0 \trianglerighteq \ldots \trianglerighteq G_n = 1$$

such that the successive quotients, $G_i/G_{i+1} \leq Z(G/G_{i+1})$. The lower central series of G is the chain of subgroups defined by $\Gamma_0(G) = G$ and $\Gamma_{n+1}(G) = [\Gamma_n(G), G]$. The upper central series is defined by letting $Z_0(G) = 1$ and $Z_n(G) = \{g \in G \mid g/Z_{n-1} \in Z(G/Z_{n-1})\}$.

It is a basic fact of group theory that a group is nilpotent if and only if the lower central series eventually reaches the identity if and ony if the upper central series reaches G.

Lemma 3.8. *Let G be a strongly connected Δ-group. Then the ascending central series is eventually constant. For n such that the series has stabilized, $G/Z_n(G)$ is centerless.*

Proof. There is some m such that for all $n \geq m$, the Δ-type of $Z_{n+2}(G)/Z_n(G)$ is strictly less than the Δ-type of G. $G/Z_n(G)$ is stronly connected, and by the previous lemma, $Z_{n+2}(G)/Z_n(G)$ is normal and central. But, then $Z_{n+1}(G) = Z_{n+2}(G)$. □

One should note that [3] proves many basic facts about strongly connected Δ-groups, including that the class is stable under quotients. So, in the previous lemma $G/Z_n(G)$ is strongly connected. Assuming less connectivity is also viable for certain applications, see [7] for a discussion and examples. More generally, quotients of n-connected Δ-groups are n-connected.

Now we can prove the partial differential analogue of Theorem 7.3.15 of [10].

Theorem 3.9. *If G is an strongly connected, solvable, nonnilpotent differential algebraic group, then G (using only the group operation) interprets an algebraically closed field of the same Δ-type as G.*

Proof. Consider $G/Z_n(G)$ such that the upper central series has stabilized (previous lemma). Since G is not nilpotnent, $G/Z_n(G)$ is a nontrivial strongly connected centerless Δ-group. Now we can apply Theorem 3.5 to get the result. □

Remark 3.10. Configuration theorems as given above in which one ends up interpreting an algebraically closed field might lead to impossibility theorems about the binding groups (definable groups of automorphisms) in the differential context.

4. Groups of typical dimension one or two

Cassidy and Singer take the approach of investigating almost simple differential algebraic groups of type one in additional detail (compared to the general case). Here, we will take what might be considered an orthogonal approach, allowing the type to be arbitrary, but assuming the typical dimension is small. The argument follows the outline of [4] or [2] with differential type and typical differential dimension replacing

Morley rank (of [4]) or U-rank (of [2]). Even though some of the arguments are similar, we reproduce them here for convenience and because while the idea of using U-rank as an analogue of Morley rank is well-established, there have been relatively fewer attempts to apply model theoretic techniques to differential algebraic ranks. Specifically, the purely group theoretic arguments are almost entirely due to [4]. The first result can be thought of as a more refined version of Reineke's theorem (in the differential setting), see section 7.2 of [10].

Proposition 4.1. *Suppose G is a Δ-group of Δ-type n. Then G has an abelian Δ-subgroup of type n.*

Proof. Consider the collection of Δ-subgroups of type n. Among this collection, choose one such connected subgroup H with minimal Kolchin polynomial. Now, assume that H is not commutative. Then take $a \in H$ which is not in the center of H. We know that $C_G(a) \cap H$ is of Δ-type less than n. This means that $H/C_G(a) \cap H$ is of Δ-type n and has the same typical Δ-dimension as H. Of course, this is also the Δ-type and typical Δ-dimension of a^H. Further, this is a generic subset of H (for a discussion of generic subsets in Δ-groups, see [7] or [6]). For any $b \in H$, a^H and b^H are either equal or disjoint. But, any two generic sets in a connected differential algebraic group intersect, the group H has only one conjugacy class which is not a singleton.

The rest of the argument would work in an arbitrary stable group, as [2] points out. Now we consider $H/Z(H)$. The action of this group on itself via conjugation is transitive, by the above arguments. Suppose that the elements are all of order 2; this forces H to be abelian, a contradiction. So, the square of any noncentral element of H is noncentral. Consider an element b such that $a = ba^{-1}b^{-1}$. Then $a = b^2ab^{-2}$. This means that $a \in C_H(b^2)$, but $a \notin C_H(b)$. Of course now we have a strictly decreasing sequence of differential algebraic groups given by $C_G(b^{2^n})$, a contradiction. ☐

Corollary 4.2. *Any almost simple differential algebraic group of typical dimension 1 is commutative.*

Lemma 4.3. *Suppose H is strongly connected and $a_\tau(H) = 2$. Then H is solvable (in two steps) or $H/Z(H)$ is simple.*

Proof. Consider the Cassidy-Singer series of H. This series is either length one or two. Suppose the series is length two. Then there is a strongly connected normal subgroup K (which is necessarily of typical dimension one). The quotient H/K is also of typical dimension one. By Proposition 4.1, we know that both K and H/K are abelian, so H is solvable.

In the case that such a K does not exist, every proper normal subgroup is of lower type. By Proposition 2.13 of [3], any normal subgroup of smaller type is central. Thus, $H/Z(H)$ is simple. ☐

Lemma 4.4. *There are no simple strongly connected differential algebraic groups with typical differential dimension 2.*

Proof. Suppose G is a counterexample. Then by Proposition 4.1 we know that there is a commutative differential algebraic subgroup of type 1. The typical dimension of this subgroup is one. Choose a minimal such subgroup, A, satisfying this criterium. Now, let $N = N_G(A)$ be the normalizer of A in G. By assumption, $N \neq G$. Now, take some $g \in N - A$. Consider $A \cap A^g$. This differential algebraic subgroup must be of lower type than G by the hypotheses regarding the choice of A. So,

$$a_\tau(A.A^g) = 2$$

If $a \in A \cap A^g$, then $C_G(a)$ contains both A and A^g, and thus their product. This is a contradiction to the strong connectedness of G, since if $C_G(a)$ contains A and A^g it must be all of G, making $Z(G)$ nontrivial and contradicting the simplicity of G. So, the intersection must be the identity.

As we vary $g \in G - N$, the sets $A.g.A$ are either equal or distinct. But $A.g.A$ contains the generic type of G, so each of these sets must be equal. So,

$$G = N \cup A.g.A$$

Claim 4.1. $G - N$ contains an involution and $N = A$.

Proof. Take some $g \in G - N$. Then, we know $x^{-1} = a_1 x a_2$ for some $a_i \in A$, since x^{-1} is not in N. Now,

$$(xa_1)^2 = a_1 a_2^{-1}$$

So, $(xa_1)^2 \in A$. Then $(xa_1)^2 = ((xa_1)^2)^{xa_1} \in A \cap A^w = 1$. So, we have our involution. Consider $K = N \cap A^{xa_1}$. We claim $N = A \ltimes K$. We know that K normalizes A, and that their intersection is the identity, so to determine that the group in question is the semidirect product, we need only to know that $N = AK$. Take $n \in N$. $nxa_1 \notin N$. But then

$$nxa_1 = a_3 x a_1 a_4$$

and $a_3^{-1} nxa_1 = a_4^{nxa_1} \in A^{nxa_1} \cap N = K$. So, $N = AK$. Next, we show that $K = 1$, which means that $N = A$. If $a \in K^{xa_1}$, then since xa_1 is an involution, $a \in A$ and $a^{xa_1} \in N$. Next we will show that for all $g \in G$, $a^g \in N$. So, take $g = a_3 x a_1 a_4$. Then $a^g = (a^{xa_1})^{a_2}$ which is in N.

Next, we will let $B = \langle a^g : g \in G \rangle$. $B \lhd G$ and $B \subseteq N$. B of the same type as G, since otherwise $G/C_G(a)$ is of type less than G, contradicting the strong connectedness of the group. By the above work A is of finite index in its normalizer, and so both $A \cap B$ and $A^{xa_1} \cap B$ are finite index in B. But then so is their intersection. Of course, this means that B must be finite, a contradiction. □

Claim 4.2. A is a maximal proper differential algebraic subgroup of G.

Proof. Let H be a differential algebraic subgroup containing A. Since G is strongly connected, $a_\tau(H) = 1$. But then for all $h \in H$,

$$a_\tau(A.h.A) \leq a_\tau(H) < 2.$$

But, this means that $h \in N_G(A) = A$. □

Claim 4.3. $G = \cup_g A^g$.

Proof. If we consider $b \in G - \cup A^g$, then $C_G(b)$ is of type one, since the conjugacy class of b is of the same type as G. Now, consider the strongly connected component of $C_G(b)$, and call this group B. Since b is not in the center of G, we know that $\tau(B) = 1$ and $a_\tau(B) = 1$. Now, B may be analysed in the same manner as A. Now, we can see that both $\cup B^g$ and $\cup A^g$ are generic subsets of G. This means that they intersect generically, which implies that $A \cap B \neq 1$. Now, for an element in the intersection, b_1, we know that $C_G(b_1)$ is equal to both A and B, which, of course, contradicts the choice of b. □

Now, take $b_1 \in G - 1$. By the previous arguments, $C_G(b_1)$ is a strongly connected abelian differential algebraic subgroup. Take an involution $w \in G - C_G(b)$. By the last claim, we can see that w is conjugate via some element in g to an element $w_1 \in C_G(b)$. From here the argument is identical to [4]:

$$w_1 w \neq w w_1$$
$$(w_1 w)^{w_1} = (w_1 w)^{-1}$$

Consider $B = C_G(w_1 w)$. Then we can see that $w_1 w \in B \cap B^{w_1}$, so we know that $B = B^{w_1}$. This means that $w \in B$. This is a contradiction, since then $w_1 w = w w_1$. □

Now we have also proved:

Theorem 4.5. *There are no nonsolvable strongly connected differential algebraic groups of typical differential dimension 2.*

5. Typical differential dimension two and nilpotentence

Cherlin's analysis [4] continues via analyzing the nonnilpotent groups of Morley rank 2. In some essential ways, that analysis uses the finiteness conditions imposed by the hypotheses of finite Morley rank. Those finiteness conditions are not available in our setting, but the hypothesis of strong connectedness is stronger than the condition of connectedness.

Theorem 5.1. *Let G be a strongly connected differential algebraic group of typical differential dimension 2 which is centerless. Then for some definable field F, G is the semidirect product of F_+ and $F^\cdot$ with $F^\cdot$ acting on F_+ via multiplication.*

Proof. By Proposition 4.1, we can find an abelian differential algebraic subgroup H of G with $\tau(H) = \tau(G)$. Further, we may assume that H is strongly connected, normal (by the results of section three), and $a_\tau(H) = 1$. In particular, this means that H is almost simple. Now consider $b \in G - C_G(H)$. By the arguments of Lemma 4.4, we know that $\tau(C_G(b)) = \tau(G)$. So, in particular, we may take T to be the (nontrivial) $\tau(G)$-connected component of $C_G(b)$. Since they are both of the same type, T is the strongly connected component of $C_G(b)$. Both H and T are strongly connected of typical differential dimension 1. So,

$$\tau(G \cap H) < \tau G.$$

Then by Lemma 3.1 of [14], we know

$$a_\tau(HT) = a_\tau(G).$$

But then HT is a closed set of differential type and typical differential dimension equal to that of G, so by the irreducibility of G, $HT = G$.

Since $Z(G) = 1$, $H \cap T = 1$. Now consider $C_G(h)$ for $h \in H - 1$. Because G is strongly connected and centerless, we know that conjugacy classes of elements of G under conjugation are of differential type $\tau(G)$. So, by the almost simplicity of H, every element $h_1 \in H$ is of the for h^t for some $t \in T$. Now suppose that $t \in C_G(h)$ for some $t \in T$. T is abelian by 4.1, so $t = t^{t_1}$ for any $t_1 \in T$. Then t centralizes h_1^t for all $t_1 \in T$, which contradicts the fact that T acts transitively on $H - 1$ via conjugation. Thus,

$$t \mapsto h^t$$

is a bijection from T to $H - 1$.

We will define addition on T via

$$x + y = z$$

if and only if

$$h^x h^y = h^z.$$

We can, of course, add a symbol to T for 0, and assume that $h^0 = 1$. This is a commutative and associative operation. We would like to prove that T is a ring under this operation and the multiplication given by the group operation. Define $-x$ to be z such that $-h^x = h^z$. We must prove that the operations are distributive, that is,

$$z(x + y) = zx + zy$$

Suppose that $x + y = z_1$. Then

$$\begin{aligned} h^{zz_1} &= zz_1hz_1^{-1}z^{-1} \\ &= z(u^x u^y z^{-1} \\ &= zu^x z^{-1} zu^y z^{-1} \\ &= u^{zx} u^{zy} \end{aligned}$$

So, $z(x + y) = zx + zy$. Now superstability theory does the rest. By [5] this ring is actually an algebraically closed field. Further, in differentially closed fields, the definable fields are actually the kernels of subsets of definable derivations (see [19]). We know, of course that $T \cup \{0\}$ and H are isomorphic as differential fields (with the definable isomorphism given above by the group operation). □

Next, as in the analysis of the previous section, we drop the condition of centerless.

Theorem 5.2. *Let G be a nonnilpotent strongly connected differential algebraic group with $a_\tau(G) = 2$. Then $G = H \ltimes T$ with $H = F_+$ and $T/Z(G) = F^\cdot$ where F is an algebraically closed (definable) field. Conjucation of H by T is given via multiplication in F.*

Proof. We know that $G/Z(G)$ satisfies the hypotheses of the previous theorem, so

$$G/Z(G) \cong F_+ \ltimes F^\cdot$$

as in that theorem. So, let H and T be the strongly connected components of the inverse images of F_+ and $F^\cdot$, respectively, under the natural quotient map. Then HT is of typical differential dimension two (see the proof in the previous theorem). So, $HT = G$. We claim that $Z(G) \cap H = 1$. Suppose not and take $h \in Z(G) \cap H$. . Then for some t_1 and t_2 in T,

$$h^{t_1} = z + h^{t_2}$$

so $t_1 t_2^{-1} \in Z(G) \cap T$, since modulo $Z(G)$ they are the same element in $F^\cdot$. But then $u^{t_1} = u^{t_2}$ so $z = 1$. □

6. Typical Δ-dimension 3

Throughout this entire section, we assume that G is a strongly connected Δ-group of typical Δ-dimension 3. The techniques are adapted from [1] and [4]. We should mention that since the main theorem in this section reduces to the case of analysis of an *almost simple* differential algebraic group, one can shorten the presentation considerably by assuming Cassidy's theorem. We have chosen not to do this for several reasons which are discussed below.

Proposition 6.1. *G is either solvable or G is almost simple.*

Proof. This is clear since a normal subgroup which witnesses non-almost simplicity means that we have the group seperated into an Abelian piece and a solvable piece by a simple dimension count and the results of the previous sections. □

Lemma 6.2. *If G has a nilpotent subgroup H with $\tau(H) = \tau(G)$ and $a_\tau(H) = 2$, then G is solvable.*

Proof. Complete this proof! □

Now G can be assumed to have a solvable nonnilpotent subgroup G_1 with $a_\tau(H) = 2$. (Prove this).

But then we can get H and T as in Theorem 5.2. Then letting T_1 be the strongly connected component of the identity of T, we can see that $G_2 = HT_1$ is a definable subgroup of G_1 (by the Δ-indecomposability theorem of [7]). In fact, G_2 is the strongly connected component of the identity in G_1. So, in what follows, simply assume G_1 was chosen to be strongly connected. Then the commutator $[G_1, G_1] = H$. and $Z(G_1) = Z(G_1) \cap H$. The proof of the following theorem roughly follows the proof of an analagous theorem in [4]. It is also similar to a proof of an analagous result in [1], which is also based on the argument in [4].

Theorem 6.3. *Let G be a nonsolvable strongly connected group of typical Δ-dimension 3. Then G is isomorphic to $SL_2(F)$ or $PSL_2(F)$.*

Proof. By the previous two lemmas, G can be assumed to have a solvable nonnilpotent subgroup G_1 with $a_\tau(G_1) = 2$. Assuming G_1 is strongly connected, along with the results of the previous section means we know $G_1 \cong H \ltimes T$ with $H \cong F_+$ for some definable field F, $Z(G_1) \subset T$, $T/Z(G_1) = F.$, and the action of $T/Z(G_1)$ on H is given by field multiplication.

Claim 6.1. The strongly connected Δ-subgroups of G_1 with Δ-type equal to $\tau(G_1)$ either contain H or are equal to T^g for some $g \in H$.

Proof. (Claim 6.1) It suffices to show that two statements:

(1) $T/Z(G_1) \cong F.$. T is a maximal proper definable subgroup of G_1.
(2) Any two definable strongly connected groups of type $\tau(G_1$ which are not equal to H are conjugate.

To prove the first statement, let $C \supseteq T$ be a Δ-subgroup of G_1. Then since $G_1 = HT$, so if $C \neq T$, there is $v \in C \cap V$ which is not the identity. By results of the previous section, any two nontrivial elements of H are conjugate by an element of T. Thus $H = v^H \cup \{1\}$. So, $H \subseteq C$ and $C = B$.

For the second statement, let $D \neq H$ be strongly connected. Then $HD = B$ by the indecomposability theorem. $Z(B) \subseteq D$, since $Z(B) \cap H = \{1\}$. We claim that $H \cap D = H \cap N_B(D) = \{1\}$. Every element of $H \cap D$ centralizes both V and D (these are abelian groups, by our analysis of groups of typical dimension 1). So, if $v \in V \cap N_B(D)$, then $[v, D] \subseteq H \cap D$, so v centralizes D. Thus v is trivial. Now note that

$$\tau(\bigcup_{v \in H} D^v - Z(B)) = \tau(G_1),$$

$$a_tau(\bigcup_{v \in V} - Z(B)) = 2$$

The above arguments also apply to T, so

$$(\bigcup_{v \in H} D^v - Z(B)) \cap (\bigcup_{v \in H} T^v - Z(B)) \neq \emptyset$$

Thus there are $v_1, v_2 \in H$ so that $T^{v_1} \cap D^{v_2} \neq Z(G_1)$. Then there is $d \in D \cap T^v) - Z(G_1)$ where $v = v_2 v_1^{-1}$. Note that $d \in C_{G_1}(T^v)$ and $d \in C_{G_1}(D)$, but $d \notin Z(G_1)$, so it must be that $T^v = D$. So, we have established the claim. □

Claim 6.2. $\forall x \notin N_G(G_1)$, $a_\tau(BxB) = 3$ and $\tau(B \cap B^x) = \tau(G)$ and $a_\tau(B \cap B^x) = 1$.

Proof. (Claim 6.2) The second statement follows from the assumptions, and the first statement follows from noting that BxB is in definable bijection with $B \cdot B^x$. □

Claim 6.3. $\forall x \notin N_G(G_1)$, $G = N_G(G_1) \cup G_1 x G_1$.

Proof. (Claim 6.3) If $x \in N_G(G_1)$ then BxB contains the generic of G. Then since two double cosets are necessarily disjoint or equal, and generic subsets can not be disjoint, $\forall x, y \in N_G(G_1)$,

$$G_1 x G_1 = G_1 y G_1$$

Now the claim follows since the union of the double cosets always covers $G - N_G(G_1)$. □

Claim 6.4. $N_G(T) \not\subset N_G(G_1)$.

Proof. (Claim 6.4) The smallest normal subgroup of G containing H is G. $\exists g \in G$ such that $H^g \not\subset G_1$. By work of the previous section, $H = G_1'$ is characteristic in G_1, so such a g is not in $N_G(G_1)$. Applying 6.3 gives $b, c \in G_1$ such that $g^{-1} = bgc$. Then we get that $H^{g^{-1}} \not\subset G_1$. So, $G_1^g \cap G_1 \neq G_1$. By 6.2,

$$\tau(G_1^g \cap G_1) = \tau(G)$$

so by 6.1 there is $v \in H$ such that $G_1^g \cap G_1 = T^v$. Thus $T^v \subset G_1^g$ so $T^{g^{-1}v} \subset B$. Also, $T^{g^{-1}v}$ cannot contain H, since otherwise T^v would contain H^g, which (as we showed above) cannot happen.

Apply 6.1 to get $u \in H$ such that

$$T^{g^{-1}v} = T^u$$

and thus

$$w = v^{-1} g u \in N_G(T)$$

then $u, v^{-1} \in G_1$ and $g \notin N_G(G_1)$, so $w \in N_G(T) - N_G(G_1)$, proving the lemma. □

Claim 6.5. $\forall w \in N_G(T) - N_G(G_1)$, $G_1 \cap G_1^w = T$ and $G = G_1 \cup G_1 w H$. The decomposition of any element of G, but not in G_1 in the form $gwu \in G_1 w H$ is unique.

Proof. (Claim 6.5) Fix $w \in N_G(T) - N_G(G_1)$. Then $T \subset G_1 \cap G_1^w$, and by 6.1, $B \cap G_1^w = T$. Then $G_1 = HT = TH$, so $G = N_G(G_1) \cup G_1 w G_1$ by 6.4. So,

$$G = N_G(G_1) \cup G_1 w H$$

Further, assume that $gwv = g'w'v' \in G_1 w H$. Then $vv'^{-1} = w^{-1} g^{-1} b' w' \in H \cap G_1^w = H \cap T = \{1\}$. Thus the decomposition is unique. □

Claim 6.6. $N_G(G_1) = G_1$ and $N_G(T) \cap G_1 = T$.

Proof. (Claim 6.6) If $b \in N_G(T)$, then $[b, T] \subset T$. Moreover $b \in G_1$ implies $[b, T \subset H = G_1'$. So, $b \in N_G(T) \cap G_1$ implies that $[b, T] = \{1\}$ and $b \in G_1 \cap C_G(T) = T$. Thus, $N_G(T) \cap G_1 = T$.

For any $c \in N_G(T) \cap N_G(G_1)$, $cw \in N_G(T) - N_G(G_1)$. Thus by 6.3,

$$cw = b^{-1}xv$$

for some $b \in G_1$ and $v \in H$. Then $T^b = T^{wv}$ is in $G_1 \cap G_1^w$ and is thus equal to T, by arguments in the proof of 6.3 and claim 6.1.

So, b and wv are in $N_G(T)$. Then $v \in N_G(T)$. But, then, by the above arguments, $b, v \in T$. But, then we know $v = 1$ and $c = b$. Thus, we see

$$N_G(T) \cap N_G(G_1) = T.$$

□

Claim 6.7. $Z(G) \subset T$

Proof. (Claim 6.7) $Z(G)$ normalizes G_1 and hence $Z(G) \subset G_1$. So, $Z(G) \subset Z(G_1) \subset T$. □

Claim 6.8. Let $w \in G$ be such that $w^2 \in T$. Then $\forall t \in T$, $t^w = t$ or for all $t \in T$, $t^w = t^{-1}$.

Proof. (Claim 6.8) Define $\sigma : T \to T$ by $\sigma(t) = t^w t$. Then $\sigma(t)^w = \sigma(t)$. Now $\tau(Ker\sigma) < \tau(G)$ implies that $im\sigma = T$ and $\sigma(t) = t$. On the other hand, if the kernel of *sigma* is fo the same type as T, then $T = ker\sigma$ and $\sigma(t) = t^{-2}$. □

The next claim is proved in [4]. It involves nothing about type or typical differential dimension. The claim depends only on the group decompositions we have set up. We reproduce the proof here for convenience.

Claim 6.9. There is $w \in N_G(T) - T$ such that $\forall t \in T$, $t^w = t^{-1}$, $w^2 \in T$, and $w^2/Z(B)$ is either -1 or 1 as an element of the interpretable field F..

Proof. (Claim 6.9) Let $x \in N_G(T) - T$ and let $b_1, b_2 \in G_1$ be such that $x^{-1} = b_1 x b_2$. $T^{b_2^{-1}} = T^{x^{-1}b_2^{-1}} = T^{b_1 x} \subseteq G_1 \cap G_1^x$. Thus, $T = T^{b_2^{-1}} = T^{b_1 x}$, so $b_1, b_2 \in T$. Then let $w = xb_1$. Then $w^2 = b_2^{-1} b_1 \in T$. Now $w \in C(T)$ or $t^w = t^{-1}$.

If $t = w^2$, then recall that $T/Z(G_1)$ is isomorphic to the multiplicative group of an algebraically closed field. So, let b be such that $b^2 + Z(G_1) = 1/(t + Z(G_1))$. Then $(wb)^2 = 1$.

Now, if $t^w = t^{-1}$ then if $t = w^2$, then $t = t^w = t^{-1}$, and so $t^2 = 1$. Then $t + Z(G_1) = \pm 1$.

Now we prove that we can take $w \notin C_G(T)$. Take $a \in H$ with $waw \notin G_1$. Then $waw = u_1 t_1 w u_2$ with

□

Claim 6.10. $G/Z(G) \cong PSL_2(F)$.

Proof. (Claim 6.10) The proof of theorem 1 of section 5 (page 23) of [4] works in this setting without modification. We have essentially constructed the *Bruhat* decomposition of the group. □

The next claim is proven by [1] and is implicitely used by [4].

Claim 6.11. $G \cong SL_2(F)$ or $PSL_2(F)$.

Proof. (Claim 6.11) We now know that G is a perfect central extension of $PSL_2(F)$. By 6.9 and 6.7 we know that $Z(G)$ is a group of exponent two. By results of [12], any perfect group G which is a central extension of $PSL_n(F)$ and is of bounded exponent is a homomorphic image of $SL_n(F)$. From this, the claim follows and G is either $SL_n(F)$ or $PSL_2(F)$. □

This also completes the proof of Theorem 6.3.

□

Remark 6.4. One can replace all but the last claim of the theorem (which needs algebraic K-theory) by Cassidy's theorem along with the classification of quasi-simple algebraic groups. Of course, the final claim uses several preliminary structure results. In fact, the entire argument can be replaced by Cassidy's theorem, and the results of [1] after noting that Cassidy's theorem implies that for this group almost simplicity is identical to indecomposability in the sense of [2]. In fact, if one is only interested in noncommutative almost simple differential algebraic groups many techniques in the present paper and [7] can be replaced by Cassidy's theorem and the machinery of superstability theory.

We chose the above method of proof for several reasons. The argument also shows how directly one can translate theorems from the superstable or finite Morley rank case to the case of differential algebraic groups when we assume strong connectedness. The analogy goes through considering the leading coefficient of the Kolchin polynomial in the same manner we consider the leading coefficient of the highest power of the ω contained in the Cantor normal form of the Lascar rank of the group. The analogy has potential for extension beyond the realm of superstability or even supersimplicity. Even in the (supersimple) case of difference-differential fields, we know that there are no bounds implied on the degree and leading coefficient of the analogue of the Kolchin polynomial in terms of the Lascar rank (for general definable sets, at least). In settings outside of differential fields, when there are multiple operators, we know of no examples of analogues of Cassidy's theorem, which is what makes the application of superstability theory possible.

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