Genericity in Differentially Closed Fields

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We work over a fixed ordinary characteristic zero differential field k, with δ a distinguished derivation. There are notions of generic points on differential varieties coming from both model theory and algebraic geometry adapted to the Kolchin topology. From the algebro-geometric perspective, the generic points on a differential variety are simply those not contained in any proper differential subvariety, (any proper Kolchin-closed subvariety) see [?]. For finite rank differential varieties, these topologically generic points are those $a \in V$ such that the field $k(a, \delta(a), \delta^2(a), \ldots)$ has transcendence degree equal to that of the differential function field of the variety (denoted $RD(V)$). When dealing with infinite rank differential varieties, the condition on the transcendence degree does not make sense as stated, but the topological notion of genericity is still valid. Rather, in that case, one should use the Kolchin polynomial. When we work with affine differential varieties in one variable, $RD(V)$ is simply equal to the highest order derivative appearing in the polynomial f, such that $V = Z(f)$. For various notions of rank and careful development of the above ideas and more, see [?].

From the model theoretic perspective, there is another notion of a generic point on a variety, $a \in V$ is generic if the Morley rank of the type $tp(a/K)$ is equal to the Morley rank of the variety. The obvious question is: to what extent do these notions of genericity agree? This question was investigated in [?].

Let us restrict our investigation to varieties in $\mathbb{A}^1$. Though this seems to be a big restriction, a result of [?] says that every differential variety embedded in projective space and of finite rank is isomorphic to a constructible set in $\mathbb{A}^1$ (that is, an open subset of a closed set in the Kolchin topology). For order 1 differential varieties, these notions of genericity are identical since algebraic dependence in a differentially closed field is equivalent to algebraic dependence in the classical sense (for fields). Already, for order 2 differential varieties the two notions of genericity are not identical. A generic point (in the Kolchin topological sense) on the differential variety

$$xx'' - x' = 0$$

has Morley rank 1 (see [?]). But, so does a generic solution to the differential subvariety $x' = 0$. Note that it is not necessary to specify which kind of generic point we speak of for the variety $x' = 0$, since both notions agree for order 1 differential subvarieties. The above example shows that model theoretic generic points are not necessarily topologically generic. All of this was pointed out by Benoist [?] in which the following question appeared: are Kolchin topological generic points always model theoretically generic? As the example above shows, irreducibility in the Kolchin topology does not imply that the variety has Morley degree 1. Before we give an example in which the topological generics are not moodel theoretic generics, in section 1, some situations where the notions agree will be noted, so that we know where not to look. There are no new or deep results in section 1. Everything there was either proved by [?] or noted by [?] (in the latter, sometimes without proof). Sections 2 and 3 contain a new example and a detailed algebraic analysis. The analysis is completely elementary differential algebra, inspired by the analysis of Poizat's example $xx'' - x' = 0$.

Kind thanks to Dave Marker for many useful conversations on these issues, support, encouragement, and for bringing to my attention the questions in [?].

1 Comparing notions of genericity

In Differential algebraic groups, the two notions agree. The proof is an exercise in stable group theory which we will do here. Suppose G is a differential algebraic group. G definable in DCF, so it is an ω-stable group. In ω-stable groups, there is a third notion of genericity. This notion is defined locally for formulas and (possibly incomplete) types. A formula $\phi(x)$ is *group generic* if finitely many translates of $\phi(x)$ by the action of the group via left multiplication cover the group. Call a type $p(x)$ group generic if each formula contain in $p(x)$ is group generic. In a ω-stable group the notions of group generic and RM-generic coincicde [?]. To the author's knowledge, the following proposition was first written down in [?].

Proposition 1.1. *Suppose that G is an irreducible differential algebraic group. Then a type is RM-generic if and only if it is a topological generic.*

Proof. Suppose that $p(x)$ is a RM-generic but not a topological generic. Then finitely many left translates of any formula in $p(x)$ cover the group, but $p(x)$ is not topological generic, so the type is contained in a proper Kolchin closed subset of G. Take the formula witnessing this, $\phi(x)$. Now, finitely many left translates of $\phi(x)$ cover the group G, and each of these is clearly closed in the Kolchin topology (if a is a topological generic in $\phi(x)$ then $g\phi(x)$ is simply the zero set of the ideal of differential polynomials vanishing at ag). But, this is a problem. Now G is the finite union of proper closed subsets.

Now, assume that $p(x)$ is a type such that any realization a is topological generic. Then take any differential polynomial $P(x)$ vanishing at a. As a is topological generic, $P(x)$ vanishes everywhere in G. So, by quantifier elimination, then only possible non-group generic formula in $p(x)$ is the negation of a differential polynomial equality. Suppose that $P(x) \neq 0$ is not group generic. Then $P(x) = 0$ is group generic, so finitely many translates cover G, which is again a contradiction if $Z(P) \cap G$ is a proper closed subset of G. Thus $P(x) \neq 0$

is group generic. Note that this argument also shows that for a differential algebraic group, irreducibility in the Kolchin topology implies that Morley degree is one. This is not true for general differential varieties. □

The last proposition is also true for homogeneous spaces in the sense of [**?**]. The proof is essentially identical.

Since for finite rank differential algebraic varieties, the notions of genericity simply come from two different notions of rank, one might look for conditions purely on the ranks of a differential algebraic variety. The first natural condition in which we expect the notions of genericity to agree is for those varieties with RM = RD. This is true for linear differential algebraic varieties, but also holds for some nonlinear differential algebraic varieties [**?**].

Proposition 1.2. *Suppose that V is a differential variety such that $RM(V) = RD(V)$. Then RM-generic $\Leftrightarrow$ topological generic.*

Proof. Suppose that a is a RM-generic point of V. Then a must lie outside all order $RD(V) - 1$ subvarieties since all of these have Morley rank at most $RD(V) - 1$, since Morley rank is always bounded by RD. Conversely suppose that a is a topological generic point of V. Then there are infinitely many order $RD(V) - 1$ subvarieties of V with Morley rank $RD(V) - 1$, since $RM(V) = RD(V)$. But then a lies outside these infinitely many subvarieties by virtue of being topological generic. So, $RM(tp(a)) \geq RD(V)$. But of course, this is the maximum that Morley rank can possibly be by assumption. So, a is RM-generic. □

Here is another simple situation where we will not find the example we are seeking.

Proposition 1.3. *Suppose that V is a differential algebraic variety such that $RD(V) = 2$. Then if $a \in V$ is topological generic, a is RM-generic.*

Proof. There are essentially two cases. Case 1: Assume that $RM(V) = 2$. In this case 1.2 applies. Case 2: Assume that $RM(V) = 1$. Any topological generic point on V is clearly not algebraic, and is thus of Morley rank at least 1. □

In light propositions 1.1, 1.2, and 1.3, if we are seeking an example in which the topological generic points are not RM-generic, we must seek a variety in which:

- Is not a group (in particular nonlinear)
- Does not have $RD(V) = RM(V)$.
- Is the zero set of a differential equation of order at least 3

Thus, a minimial rank example in which the topological generics are RM-generic would be a third order differential variety, V, with only finitely many order two differential subvarieties of Morley rank two, $\{W_i\}_{i=1}^n$, such that the constructible set $V - \cup_{i=1}^n W_i$ has Morley rank 1. Verifying both of these conditions for a given example basically involves proving restrictions on the possible order 2 and order 1 subvarieties. For examples, see the exposition of Poizat's example in [**?**].

For the remainder of the paper, we will let $f(x) = xx''' - x''$ and $V = Z(f)$. V has an order 2 subvariety, $Z(x'')$. In fact this is the only order 2 subvariety and $V - Z(x'')$ is strongly minimal. The following two sections are devoted to proving these facts by analyzing the subvarieties of V.

2 Order 2 Subvarieties

Throughout, we let $f(x) = xx''' - x''$ and $V = Z(f)$. This has the obvious order 2 subvariety $Z(x'')$. We will show that this is the only order 2 subvariety. So, let $g \in K[x, x, x'']$ be an order 2 diffrential polynomial. That is,

$$g = \sum_{n=0}^{N} a_n (x'')^n$$

where $a_n \in K[x, x']$, $N > 0$, and $a_N \neq 0$. In analyzing the order two differential subvarieties of V, it is only necessary to consider the zero sets of single differential polynomials, as every Kolchin closed subset of $\mathbb{A}^1$ is the zero set of a single differential polynomial. If $f \in I(g)$, so is any differential polynomial g_1 which differs from $xD(g)$ by a multiple of f. Or, if you like, think of f as a relation which holds on the differential polynomials in $I(g)$.

$$xD(g) = x \sum_{n=0}^{N} (a_n^D + \frac{\partial a_n}{\partial x} x' + \frac{\partial a_n}{\partial x'} x'')(x'')^n + x \sum_{n=0}^{N} n a_n (x'')^{n-1} x'''$$

But, modulo f,

$$xD(g) \equiv_f g_1 := x \sum_{n=0}^{N} (a_n^D + \frac{\partial a_n}{\partial x} x' + \frac{\partial a_n}{\partial x'} x'')(x'')^n + \sum_{n=0}^{N} n a_n (x'')^{n-1} x''$$

Now, unlike $xD(g)$, the new differential polynomial, g_1 is order 2. So, if it is to be in $I(g)$, then it must be the case that g divides g_1. The argument will proceed by considering x'' degree.

The leading term (with respect to x'') of g_1 is $x \frac{\partial a_N}{\partial x'} (x'')^{N+1}$. But the leading term of g is a_N, which has higher x' degree, so there is no chance that g divides g_1 unless $\frac{\partial a_N}{\partial x'} = 0$. So, $a_N \in K[x]$. Now, assuming that $\frac{\partial a_N}{\partial x'} = 0$, the leading term of g_1 is

$$(x a_N^D + x x' \frac{\partial a_n}{\partial x} + x \frac{\partial a_{N-1}}{\partial x'} + N a_N)(x'')^N \tag{1}$$

So, since the leading term of g is $a_N(x'')^N$ and the x'' degree of the polynomials is the same, using (1), one can see

$$g_1 = g(x\frac{a_N^D + x'\frac{\partial a_n}{\partial x} + \frac{\partial a_{N-1}}{\partial x'}}{a_N} + N) \tag{2}$$

Specifically, by previous work we know the following:

$$\begin{aligned} a_N &= \sum_{k=0}^{m} b_k x^k \\ \frac{\partial a_N}{\partial x} &= \sum_{k=0}^{m} k b_k x^{k-1} \\ a_N^D &= \sum_{k=0}^{m} D(b_k) x^k \end{aligned}$$

Now we can compare the $(x'')^0$ terms on either side of the equation (2).

$$LHS = xa_o^D + xx'\frac{\partial a_0}{\partial x} \tag{3}$$

$$RHS = a_0(x\frac{a_N^D + x'\frac{\partial a_n}{\partial x} + \frac{\partial a_{N-1}}{\partial x'}}{a_N} + N) \tag{4}$$

Now, by comparing the x' leading terms of (3) and (4) one can see

$$\frac{\partial a_{N-1}}{\partial x'} = c + x'd \tag{5}$$

where $c, d \in K[x]$. Now (2) becomes somewhat simpler,

$$g_1 = g(x\frac{a_N^D + x'\frac{\partial a_n}{\partial x} + c + x'd}{a_N} + N) \tag{6}$$

After regrouping some terms, this reduces (3) and (4) to

$$LHS = xa_0^D + xx'\frac{\partial a_0}{\partial x} \tag{7}$$

$$RHS = a_0(N + x\frac{a_N^D + c}{a_N} + xx'\frac{\sum_{k=0}^{m} k b_k x^{k-1} + d}{a_N}) \tag{8}$$

So, let

$$a_0 = \sum_{i=0}^{m_1} c_i (x')^i$$

where $c_i \in K[x]$. The x' leading term from (8) is

$$c_{m_1}(x')^{m_1+1} x\frac{\sum_{k=0}^{m} k b_k x^{k-1} + d}{a_N} \tag{9}$$

And the x' leading term from (7) is

$$x\frac{\partial c_{m_1}}{\partial x}(x')^{m_1+1} \tag{10}$$

So, if we suppose that

$$c_{m_1} = \sum_{j=1}^{m_2} d_j x^j$$

then from (9) and (10) we have the requirement:

$$d_{m_2} x^{m_2} x(\frac{\sum_{k=0}^{m} kb_k x^{k-1} + d}{a_N}) = m_2 d_{m_2} x^{m_2}$$

Thus,

$$x\frac{\sum_{k=0}^{m} kb_k x^{k-1} + d}{a_N} = m_2 \tag{11}$$

So, we compare the lower order x terms, and see that $d_k = 0$ for $k < m_2$ But further, now we have

$$g_1 = g(N + x\frac{a_N^D + c}{a_N} + x'm_2). \tag{12}$$

Consider the $(x')^0$ term of the $(x'')^0$ term. Now, let

$$c_0 = \sum_{k=0}^{m_3} \alpha_k x^k.$$

Next, we may assume that $\alpha_{m_3} = 1$. If note, then divide the original polynomial by α_{m_3}; it still generates the same zero set. Comparing the left and right sides of (12).

$$LHS = xc_0^D \tag{13}$$

$$RHS = c_0(N + x\frac{a_N^D + c}{a_N}) \tag{14}$$

So,

$$xc_0^D = c_0(N + x\frac{a_N^D + c}{a_N}) \tag{15}$$

$$\sum_{k=0}^{m_3-1} \alpha_k^D x^{k+1} = \sum_{k=0}^{m_3} \alpha_k x^k (N + x\frac{a_N^D + c}{a_N}) \tag{16}$$

By comparing the x leading term of (16), we can see

$$x\frac{a_N^D + c}{a_N} \in K.$$

But, now the RHS of (16) has a nonzero x^j term where j is the minimum integer such that $\alpha_j \neq 0$. But the LHS of (16) has no x^j term. So, it must be the case that $c_0 = 0$.

Now we proceed by induction on the number of c_i which are zero (we have just proved the base case of the induction). So, suppose the $c_0, c_1, \dots c_l$ are all zero. Consider the $(x')^{l+1}$ terms in the $(x'')^0$ term,

$$g_1 = g(N + x\frac{a_n^D + c}{a_N} + x'm_2) \tag{17}$$

On the LHS (17) (recall that $c_l = 0$ so $\frac{\partial a_0}{\partial x} xx'$ contributes no $(x')^{l+1}$ terms):

$$xc_{l+1}^D.$$

On the RHS of (17) ($c_l = 0$ so $x'M_2$ does not give any $(x')^{l+1}$ terms):

$$c_{l+1}(N + x\frac{a_n^D + c}{a_N}).$$

But, this is precisely the condition that we showed was impossible unless $c_l = 0$ (the statement is literally the same but now with c_{l+1} instead of c_0). But, now c_i must be zero for $i = 1 \dots m_1$, that is x'' divides g. But, since we assumed that g is irreducible, $g = x''$. Thus, $x'' = 0$ is the unique irreducible order two subvariety of V.

Note, at this point we know that $RU(V) = RM(V) = 2$. To see this, recall that $\{Z(x' = c)\}_{c \in C_K}$ is a uniformly definable family of order 1 subvarieties of $Z(x'')$. On the other hand $RH(V) = RD(V) = 3$. The only remaining question about ranks associated with this differential variety is the rank of the Kolchin open subvariety $V - Z(x'')$. At this point, it might be the case that there is a uniformly definable family of order 1 subvarieties, making topological generic points of V Lascar and Morley Rank 2 (it might, a priori, be the case that the Lascar rank and Morley rank differ). So, the remaining questions about the rank of this variety can be answered by understanding the order 1 subvarieties which are outside of $Z(x'')$. Actually, in some sense any answer would be interesting. If there were infinitely many order 1 subvarieties of this open subvariety, but not an infinite uniformly definable family, then it would be an example of a definable set for which Morley rank and Lascar rank differ (previous examples have had RM at least 5 [**?**]). If Morley rank and Lascar rank were both two; perhaps the situation is not quite so interesting, but it is another example of an irreducible variety (in the Kolchin topology) having Morley degree two. Next, we analyze the order one subvarieties of V.

3 Order 1 Subvarieties

Now consider the potential order 1 subvarieties of $V = Z(xx''' - x'')$. Throughout, let $f(x) = xx''' - x''$.

Any such subvariety is the zero set of an irreducible differential polynomial $g(x, x') \in K[x, x']$. So, let

$$g(x, x') = \sum_{n=0}^{N} a_n (x')^n,$$

where $a_n \in K[x]$ and $a_N \neq 0$. Now , we wish to restrict the types of differential polynomials which might occur as order one subvarieties, so the general technique will be to differentiate twice, and apply the third order relation which holds on V.

$$D(g) = \sum_{n=0}^{N} n a_n (x')^{n-1} x'' + \sum_{n=0}^{N} \frac{\partial a_n}{\partial x} (x')^{n+1} + \sum_{n=0}^{N} a_n^D (x')^n$$

$$\begin{aligned}
D^2(g) = &\sum_{n=0}^{N} n a_n (x')^{n-1} x''' + \sum_{n=0}^{N} n \frac{\partial a_n}{\partial x} (x')^n x'' + \sum_{n=0}^{N} n(n-1) a_n (x')^{n-2} (x'')^2 \\
&+ \sum_{n=0}^{N} n a_n^D (x')^{n-1} x'' + \sum_{n=0}^{N} \frac{\partial a_n^D}{\partial x} (x')^{n+1} + \sum_{n=0}^{N} \frac{\partial^2 a_n}{\partial x^2} (x')^{n+2} \\
&+ \sum_{n=0}^{N} (n+1) \frac{\partial a_n}{\partial x} (x')^n x'' + \sum_{n=0}^{N} a_n^{D^2} (x')^n + \sum_{n=0}^{N} \frac{\partial a_n^D}{\partial x} (x')^{n+1} \\
&+ \sum_{n=0}^{N} n a_n^D (x')^{n-1} x''
\end{aligned}$$

Now, multiply both sides of the equation by x, and note that since we assume that $f \in I(g)$, we know that we might, given a differential polynomial in $I(g)$, we might replace any instance of xx''' with x'' and we would still have a differential polynomial in $I(g)$. So, multiply the above expression for $D^2(g)$ by x and replace the instance of xx''' by x''. Now we have some other differential polynomial, call it $g_1(x)$ which is still in $I(g)$.

$$g_1(x) = \sum_{n=0}^{N} n a_n (x')^{n-1} x'' + x \sum_{n=0}^{N} n \frac{\partial a_n}{\partial x} (x')^n x'' + x \sum_{n=0}^{N} n(n-1) a_n (x')^{n-2} (x'')^2 \tag{18}$$

$$+ x \sum_{n=0}^{N} n a_n^D (x')^{n-1} x'' + x \sum_{n=0}^{N} \frac{\partial a_n^D}{\partial x} (x')^{n+1} + x \sum_{n=0}^{N} \frac{\partial^2 a_n}{\partial x^2} (x')^{n+2} \tag{19}$$

$$+ x \sum_{n=0}^{N} (n+1) \frac{\partial a_n}{\partial x} (x')^n x'' + x \sum_{n=0}^{N} a_n^{D^2} (x')^n + x \sum_{n=0}^{N} \frac{\partial a_n^D}{\partial x} (x')^{n+1} \tag{20}$$

$$+ x \sum_{n=0}^{N} n a_n^D (x')^{n-1} x'' \tag{21}$$

Now both g_1 and $D(g)$ are differential polynomials in $I(g)$. So, we could replace instances of

$$\sum_{n=0}^{N} na_n(x')^{n-1}x''$$

in $g_1(x)$ with

$$-\sum_{n=0}^{N} \frac{\partial a_n}{\partial x}(x')^{n+1} - \sum_{n=0}^{N} a_n^D(x')^n$$

and get another differential polynomial, call it g_2, in $I(g)$ (since on the variety $Z(g)$, this relation holds).

$$\begin{aligned}
g_2(x) = &-\sum_{n=0}^{N} \frac{\partial a_n}{\partial x}(x')^{n+1} - \sum_{n=0}^{N} a_n^D(x')^n && (22)\\
&+ xx'(-\sum_{n=0}^{N} \frac{\partial^2 a_n}{\partial x^2}(x')^{n+1} - \sum_{n=0}^{N} \frac{\partial a^D}{\partial x}{}_n(x')^n) && (23)\\
&+ x(-\sum_{n=0}^{N}(n+1)\frac{\partial a_n}{\partial x}(x')^n x'' - \sum_{n=0}^{N} na_n^D(x')^{n-1}x'') && (24)\\
&+ x(-\sum_{n=0}^{N} \frac{\partial a_n^D}{\partial x}(x')^{n+1} - \sum_{n=0}^{N} a_n^{D^2}(x')^n) && (25)\\
&+ x\sum_{n=0}^{N} \frac{\partial a_n^D}{\partial x}(x')^{n+1} + x\sum_{n=0}^{N} \frac{\partial^2 a_n}{\partial x^2}(x')^{n+2} && (26)\\
&+ x(-\sum_{n=0}^{N} \frac{\partial^2 a_n}{\partial x^2}(x')^{n+2} - \sum_{n=0}^{N} \frac{\partial a_n^D}{\partial x}(x')^{n+1}) && (27)\\
&+ x\sum_{n=0}^{N} a_n^{D^2}(x')^n + x\sum_{n=0}^{N} \frac{\partial a_n^D}{\partial x}(x')^{n+1} && (28)\\
&+ x(-\sum_{n=0}^{N} \frac{\partial a_n^D}{\partial x}(x')^{n+1} - \sum_{n=0}^{N} a_n^{D^2}(x')^n) && (29)
\end{aligned}$$

Note that $x'g(x) \in I(g)$ so $D(x'g(x)) \in I(g)$ so the following relation holds on $Z(g)$,

$$\sum_{n=0}^{N}(n+1)a_n(x')^n x'' = -\sum_{n=0}^{N} \frac{\partial a_n}{\partial x}(x')^{n+2} - \sum_{n=0}^{N} a_n^D(x')^{n+1}.$$

Taking the partial derivative with respect to x yields an identity used in line 3 of the equation for g_1.

Also note that

$$\frac{\partial}{\partial x'}(\sum_{n=0}^{N} na_n(x')^{n-1}(x'')^2) = \sum_{n=0}^{N} n(n-1)a_n(x')^{n-2}(x'')^2.$$

This means that on the variety $Z(g)$ the following relation (obtained by differentiating both sides of (**) with respect to x' and multiplying by xx") holds:

$$x\sum_{n=0}^{N} n(n-1)a_n(x')^{n-2}(x'')^2 = x(-\sum_{n=0}^{N}(n+1)\frac{\partial a_n}{\partial x}(x')^n x'' - \sum_{n=0}^{N} na_n^D(x')^{n-1}x'').$$

This explains line (3) in the expression for $g_2(x)$ above.

Now, there are still two instances of x" in $g_2(x)$, namely, from line (3) of the expression for g_2:

$$x(-\sum_{n=0}^{N}(n+1)\frac{\partial a_n}{\partial x}(x')^n x'' - \sum_{n=0}^{N} na_n^D(x')^{n-1}x'').$$

Using the same technique as above, get rid of these instances via a relation which holds on $Z(g)$ to obtain the following differential polynomial, which is in $I(g)$:

$$
\begin{aligned}
g_3(x) = &-\sum_{n=0}^{N}\frac{\partial a_n}{\partial x}(x')^{n+1} - \sum_{n=0}^{N} a_n^D (x')^n \\
&+ xx'(-\sum_{n=0}^{N}\frac{\partial^2 a_n}{\partial x^2}(x')^{n+1} - \sum_{n=0}^{N}\frac{\partial a^D}{\partial x}{}_n(x')^n) \\
&+ x(\sum_{n=0}^{N}\frac{\partial^2 a_n}{\partial x^2}(x')^{n+2} + \sum_{n=0}^{N}\frac{\partial a_n^D}{\partial x}(x')^{n+1}) \\
&+ x(\sum_{n=0}^{N}\frac{\partial a_n^D}{\partial x}(x')^{n+1} + \sum_{n=0}^{N} a_n^{D^2}(x')^n) \\
&+ x(-\sum_{n=0}^{N}\frac{\partial a_n^D}{\partial x}(x')^{n+1} - \sum_{n=0}^{N} a_n^{D^2}(x')^n) \\
&+ x\sum_{n=0}^{N}\frac{\partial a_n^D}{\partial x}(x')^{n+1} + x\sum_{n=0}^{N}\frac{\partial^2 a_n}{\partial x^2}(x')^{n+2} \\
&+ x(-\sum_{n=0}^{N}\frac{\partial^2 a_n}{\partial x^2}(x')^{n+2} - \sum_{n=0}^{N}\frac{\partial a_n^D}{\partial x}(x')^{n+1}) \\
&+ x\sum_{n=0}^{N} a_n^{D^2}(x')^n + x\sum_{n=0}^{N}\frac{\partial a_n^D}{\partial x}(x')^{n+1} \\
&+ x(-\sum_{n=0}^{N}\frac{\partial a_n^D}{\partial x}(x')^{n+1} - \sum_{n=0}^{N} a_n^{D^2}(x')^n)
\end{aligned}
$$

Combining like terms, notice that all of the lines in pairs after the first. Then multiplying by minus 1:

$$
g_4(x) = \sum_{n=0}^{N}\frac{\partial a_n}{\partial x}(x')^{n+1} + \sum_{n=0}^{N} a_n^D (x')^n
$$

Now g_4 is an order 1 differental polynomial contained in $I(g)$, and we notice that

$$
g_5 = D(g) - g_4 = (\sum_{n=0}^{N} n a_n (x')^{n-1})x''.
$$

Of course g_5 is still in $I(g)$, but then either $x'' \in I(g)$ or

$$
\sum_{n=0}^{N} n a_n (x')^{n-1} \in I(g).
$$

But, the latter is impossible since the differential polynomial is degree 1 and thus the only chance for it to be in $I(g)$ is by virtue of being divisible by g.

By x' degree it is impossible that g divides $\sum_{n=0}^{N} na_n(x')^{n-1} \in I(g)$. Now we know $x'' \in I(g)$. So, the only order 1 subvarieties of V are actually subvarieties of $x'' = 0$.