Name: ______________________________

Student ID: ______________________________

Your submission must consist only of the pages given.

1. (4 points) True or False (Fill in the blank with T or F)
   (a) If A is an $n \times n$ matrix such that there is $b \in \mathbb{R}^n$ for which $Ax = b$ does not have a unique solution, then we can still apply Cramer's rule to get at least one of the solutions. ☐
   (b) If A is an $n \times n$ matrix, then $Null(A) \cap Col(A) = \{0\}$. ☐
   (c) Any real $n \times n$ matrix A where n is odd has a nonzero real eigenvector. ☐
   (d) There is an $n \times n$ real matrix with $n - 1$ distinct real eigenvalues and 1 complex eigenvalue with nonzero imaginary part. ☐

2. (4 points) Multiple choice: If A is a 2×2 matrix such that $A^2 = I_2$ where I_2 is the 2×2 identity matrix, which of the following could be the characteristic polynomial of A? Mark an "x" in the box next to the correct answers:
   1. $(x-1)(x-1)(x+1)$ ☐
   2. $x^2 - x$ ☐
   3. $x^2 - 2x + 1$ ☐
   4. $x^2 - 1$ ☐

3. (4 points) Multiple choice: Which of the following pairs of matrices are similar? Mark an "x" in the box next to the correct answers:
   1. $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 0 & 2 \end{pmatrix}$ ☐
   2. $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$ ☐
   3. $\begin{pmatrix} 3 & 3 \\ 1 & 5 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & -6 \end{pmatrix}$ ☐
   4. $\begin{pmatrix} 3 & -1 \\ 1 & 3 \end{pmatrix}, \begin{pmatrix} 5 & 3 \\ 1 & 1 \end{pmatrix}$ ☐

4. (6 points) An $n \times n$ matrix O is called *orthogonal* if $O^T O = I_n$ where I_n is the $n \times n$ identity matrix. Find a 4×4 orthogonal matrix whose first column is

$$\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

Additional space for problem 4:

5. (6 points) Let P_2 be the set of polynomials of degree ≤ 2. Consider the linear map $T : P_2 \to P_2$ given by $T(p)(x) = 2p(x) - p'(x)x$ where p' denotes the derivative of the polynomial p.

   1. Show that T is linear (you can assume without proof that $(p_1 + p_2)'(x) = p_1'(x) + p_2'(x)$ where $p_1, p_2 \in P_2$).
   2. Find the eigenvectors and eigenvalues of T.

Additional space for problem 5:

6. (6 points) Find the determinant of

$$A = \begin{pmatrix} a & b & b & b & b \\ b & a & b & b & b \\ b & b & a & b & b \\ b & b & b & a & b \\ b & b & b & b & a \end{pmatrix}$$

where a and b are any real numbers (Hint: Use row operations to cancel many elements and simplify the calculation).

Additional space for problem 6:

7. (2 points) (Bonus) Show that if an orthogonal matrix O has determinant -1, then -1 is an eigenvalue of O.

Additional space for problem 7: