Math 250A
Professor Kenneth A. Ribet
First Midterm Exam September 29, 2020
8:10–9:30AM

---

You have 80 minutes to work on the exam and 15 minutes to upload your work to `Gradescope`. You may consult the textbook, all the material on `bCourses`, the class `piazza` and your own notes. In case of questions, post a private note to instructors on `piazza`. Any clarifications or corrections that need to be promulgated will be added to a pinned post on `piazza`.

**Not permitted: online searches, other uses of the internet, collaboration with other people (electronic or otherwise).** Please act with honesty, integrity and respect for others. At the conclusion of the exam, please copy and sign this statement:

"*As a member of the UC Berkeley community, I acted with honesty, integrity, and respect for others during this exam. The work that I am uploading is my own work. I did not collaborate with or contact anyone during the exam. I did not obtain solutions from* `chegg.com` *or other sites. I adhered to all instructions for this examination.*"

**1.** Let $K$ be the field of integers mod $p$, where $p$ is a prime number. Compute the number of elements in the set

$$\left\{ g \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} g^{-1} \,|\, g \in \mathbf{GL}(2, K) \right\}.$$

**2. a.** Suppose that a finite group $G$ acts on a finite set $X$. By definition, for $g \in G$, the *orbits of* $g$ acting on $X$ are the orbits in $X$ of the subgroup of $G$ generated by $g$. If the distinct orbits of $g$ on $X$ have sizes $\ell_1$, $\ell_2, \ldots$, $\ell_k$, explain briefly why the sign of the permutation $x \mapsto gx$ of $X$ is $(-1)^{\ell_1 - 1} \cdots (-1)^{\ell_k - 1}$.

**b.** Let $G$ now act on itself by left multiplication: $gx$ is the indicated group product of $g$ and $x$ (and $X = G$). Show that these statements are equivalent: (1) there is an element of $G$ that induces an odd permutation of $G$; (2) $G$ has even order and has an element whose order is the largest power of 2 dividing $|G|$. [A suave rephrasing of (2): $G$ has 2-Sylow subgroups, and these subgroups are cyclic.]

**c.** Assume that the order of $G$ is an even integer bigger than 2. If $G$ does have an element whose order is the largest power of 2 dividing $|G|$, show that it is not a simple group.

**3.** Suppose that $p$ and $q$ are prime numbers with $p < q$. Show that all groups of order $pq$ are cyclic if and only if $q \not\equiv 1 \bmod p$.

**4.** Let $G$ be a group, and suppose that $H$ is a finite normal subgroup of $G$. Assume that $P$ is a $p$-Sylow subgroup of $H$ (where $p$ is a prime number). Show that every element of $G$ may be written as the product of an element of $H$ and an element of $G$ that normalizes $P$.