Math 250A, Fall 2004
Last Midterm Exam—November 4, 2004

Please put away all books, calculators, electronic games, cell phones, pagers, .mp3 players, PDAs, and other electronic devices. You may refer to a single 2-sided sheet of notes. Explain your answers in full English sentences as is customary and appropriate. Your paper is your ambassador when it is graded.

All rings are rings with identity!

1. *Suppose that I and J are ideals in a commutative ring R such that $I + J = R$. Establish the surjectivity of the natural map $R \to R/I \times R/J$, $\quad r \mapsto (r \bmod I, r \bmod J)$. (Don't just name a theorem; write down a complete proof.)*

This is the Chinese Remainder Theorem, but we are asked to supply a proof. If $I + J = R$, then 1 is an element of $I + J$, so that there are $x \in I$, $y \in J$ with $x + y = 1$. Given $a, b \in R$, we can write down $r := ay + bx$ and see that r has the same image as a mod I and the same image as b mod J.

If $I + J = R$, as above, show that $I^n + J^m = R$ whenever n and m are positive integers.

We can take $m = n$ since $I^n + J^m$ contains $I^n + J^n$ if $n \geq m$. Suppose that $1 = x + y$ as above. Then $1 = (x + y)^{2n}$. If you expand out $(x + y)^{2n}$ by the binomial theorem, you'll see that each term is divisible either by x^n or by y^n. Hence $(x + y)^{2n}$ lies in $I^n + J^n$.

2. *Let k be a field, and let V be the k-vector space consisting of $(a_1, a_2, \ldots)$ with $a_i \in k$ and a_i nonzero only for a finite set of i. Let $R = \operatorname{End}_k V$ be the ring of linear transformations $V \to V$; the ring multiplication is composition. (If V were instead the smaller vector space k^n, then R would be the ring of $n \times n$ matrices over k.)*

View R as a (left) R-module under the ring product and V as an R-module under the natural operation $r \cdot v = r(v)$. Prove that the R-module R is isomorphic to the product $\prod_{i=1}^{\infty} V$ and deduce that the R-modules R and $R \oplus R$ are isomorphic.

We have $R = \operatorname{End}_k V = \operatorname{Hom}_k(V, V) = \operatorname{Hom}_k(\bigoplus_{i=1}^{\infty} k, V) = \prod_{i=1}^{\infty} \operatorname{Hom}_k(k, V) = \prod_{i=1}^{\infty} V$. In this string in equalities, we have used that V is the direct sum of a countable number of copies of k, that maps from a direct sum amount to maps from each of the summands, and that a k-linear map from k to V is the same thing as a vector in V. The map $f : R \xrightarrow{\sim} \prod V$ from the left side to the right side takes $r \in R$ to $(re_1, re_2, \ldots)$, where the e_i are the standard basis vectors of V. (For example, $e_1 = (1, 0, 0, \ldots)$.) If a is in R, then $f(ar) = (are_1, are_2, \ldots) = a(re_1, re_2, \ldots)$, so the map f is R-equivariant. (This means that it's a map of R-modules.) The reason that R and $R \oplus R$ are then isomorphic is that the R-modules $\prod_{i=1}^{\infty} V$ and $(\prod_{i=1}^{\infty} V) \times (\prod_{i=1}^{\infty} V)$ are clearly the same thing: the index sets $\{1, 2, 3, \ldots\}$ and $\{1, 2, 3, \ldots\} \coprod \{1, 2, 3, \ldots\}$ are in bijection.

3. *In each situation, either give a concrete example of the object described or else explain why there cannot be an example. Include explanations!*

A finitely generated torsion free module, over a principal ideal domain, that is not a free module.

We proved in class that finitely generated modules over PIDs are free modules. This is basically all that I want you to say. If you give some details about the proof, I'll think good thoughts about you. If you supply false details about the proof, I'll be sad.

A torsion-free abelian group that is not a free abelian group.

The example that I had in mind was $\mathbf{Q}$. By coincidence, or not, one of the students came in to my office this morning (Thursday) to ask why $\mathbf{Q}$ is not free. Here's one reason: if F is the free abelian group on a set I and A is an abelian group, then $\mathrm{Hom}(F, A)$ is (in natural bijection with) the set of functions from I to (the underlying set of) A. Take $F = \mathbf{Q}$ and $A = \mathbf{Z}$. The group $\mathrm{Hom}(\mathbf{Q}, \mathbf{Z})$ is the trivial group (0) (as you can see in various ways). This means that there is only one map of sets $I \to \mathbf{Z}$; we can infer from this that $I =$, the empty set. Of course, the free abelian group on the empty set is (0); since $\mathbf{Q}$ is non-zero, we have a contradiction.

A unique factorization domain that is not a principal ideal domain.

This is a fair question, I think. A standard example is the polynomial ring $K[x, y]$, where K is a field. We will know this "officially" only next week, but I'm sure that you've seen this example, or other examples, before. (My example is mentioned on page 113 of the text.) In $K[x, y]$, the ideal (x, y) of polynomials with no constant term is not a principal ideal. Indeed, there is no polynomial $\neq 1$ that divides both x and y.

An exact sequence of $\mathbf{Z}$*-modules that does not split.*

We've seen examples in class. One that comes to mind is

$$0 \to 2\mathbf{Z} \to \mathbf{Z} \to \mathbf{Z}/2\mathbf{Z} \to 0,$$

where the map $2\mathbf{Z} \to \mathbf{Z}$ is the inclusion. It is clear that this sequence does not split because $\mathbf{Z}$ contains no submodule isomorphic to $\mathbf{Z}/2\mathbf{Z}$.

4. *Let A be an integral domain, and let K be the quotient field of A. Suppose that M and N are A-submodules of K such that $MN = A$. (Recall that MN is the smallest A-submodule of K that contains all products mn with $m \in M, n \in N$.) Show that the A-modules M and N are finitely generated.*

This is an abstraction of the first homework problem about Dedekind domains. Since $1 \in MN$, we can write 1 as a finite sum $\sum m_i n_i$. If m is in M, we have $m = m \cdot 1 = m \sum m_i n_i = \sum (mn_i) m_i$, which is a linear combination of the m_i with coefficients in A. Hence M is generated by the m_i; similarly for N.

Let A be a commutative ring, and let M be an A-module. For each prime ideal $\mathfrak{p}$ of A, let $M_\mathfrak{p}$ be the localization of M at $\mathfrak{p}$ and let $\iota_\mathfrak{p} : M \to M_\mathfrak{p}$ be the natural map (which takes $m \in M$ to the formal fraction $\dfrac{m}{1}$). If m is a non-zero element of M, show that there is a maximal ideal $\mathfrak{p}$ such that $\iota_\mathfrak{p}(m)$ is non-zero.

The quantity $\iota_\mathfrak{p}(m)$ is zero if and only if m is annihilated by some element of the multiplicative set $A \setminus \mathfrak{p}$. Hence $\iota_\mathfrak{p}(m)$ is non-zero if and only if all elements of A that annihilate m are contained in $\mathfrak{p}$. The set $\mathrm{Ann}(m) = \{ a \in A \mid am = 0 \}$ is an ideal of A. It is a proper ideal because m is non-zero. We have seen that every proper ideal of a commutative ring is contained in a maximal ideal if the ring. If we take $\mathfrak{p}$ to be a maximal ideal of A that contains $\mathrm{Ann}(m)$, then $\iota_\mathfrak{p}(m)$ will be non-zero, as required.