MATH 110 - FINAL

LECTURE 1, SUMMER 2009

August 13, 2009

Name: ____

1. (10 points) Give two equivalent definitions (or characterizations) of each of the following.

(a) A normal operator on an inner-product space V.

(b) A generalized eigenvector of an operator T.

(c) A positive operator on an inner-product space V.

(d) An isometry on an inner-product space V.

2. (15 points) Give examples, with brief justification, of each of the following.

(a) An operator on $\mathbb{R}^2$ which is not self-adjoint with respect to the standard inner product.

(b) An isometry on $\mathbb{R}^4$ with no (real) eigenvalues.

(c) An operator on $\mathbb{C}^4$ whose characteristic polynomial equals the square of its minimal polynomial.

3. (20 points) Suppose that P is an operator on an inner-product space V such that $P^2 = P$. Prove that P is an orthogonal projection if and only if it is self-adjoint.

4. (20 points) Suppose that T is a self-adjoint operator on a inner-product space V such that there exists $v \in V$ with $\|v\| = 1$ such that $\langle Tv, v \rangle > 1$. Prove that there exists an eigenvalue of T which is larger than 1. (Hint: Spectral Theorem)

5. (20 points) Let V be a complex vector space. If you get stuck on part (a) below, assume it is true and use it in part (b).

(a) Prove that if N is a nilpotent operator on V, then $N + I$ has a square root.

(b) Prove that any invertible operator T on V has a square root. (Hint: Use the generalized eigenspaces of T)

6. (15 points) Suppose that an operator T on a complex vector space has characteristic polynomial $z^3(z-2)^5(z+1)^2$ and minimal polynomial of the form

$$z^2(z-2)^k(z+1)^\ell \text{ where } k > 2 \text{ and } \ell \geq 1.$$

Suppose further that $\dim \operatorname{range}(T - 2I) = 7$ and that the eigenspace corresponding to -1 is 1-dimensional. Find, with justification, the Jordan blocks which make up the Jordan form of T. You do not have to write out the full Jordan form itself.

7. (0 points) Draw a picture portraying your love of linear algebra.