UC Berkeley Math 10B, Spring 2015: Midterm 1

Prof. Sturmfels, February 26; **SOLUTIONS**

1. (5 points) Find the coefficient of x^6y^4 in $\left(2x + \frac{1}{2}y\right)^{10}$.

The Binomial Theorem tells us that

$$(a+b)^{10} = \sum_{i=0}^{10} \binom{10}{i} a^i b^{10-i}$$

If we set $a = 2x$ and $b = (1/2)y$ then this implies

$$(2x + \frac{1}{2}y)^{10} = \sum_{i=0}^{10} \binom{10}{i} (2x)^i (y/2)^{10-i} = \sum_{i=0}^{10} \binom{10}{i} 2^{2i-10} \cdot x^i y^{10-i}.$$

The desired coefficient appears for $i = 6$. It equals

$$\binom{10}{4} \cdot 2^2 = 210 \cdot 4 = \mathbf{840}.$$

2. (5 points) We toss a biased coin, where the probability that heads comes up is 0.4. What is the expected number of heads that come up when it is flipped 20 times?

This is analogous to Problem 11 in Homework 5. Let X be the number of heads. This is a a binomial random variable with $p = 0.4$ and $n = 20$. Its expected value equals $E[X] = np = 20 \cdot 0.4 = 8$. Hence the expected number of heads is **8**.

3. (5 points) Seven people stand in line to greet the President of the United States. However, one of the seven people is nervous and does not want to be the first person in line. In how many different ways can the seven people stand?

There are $7! = 5040$ ways for seven people to stand in line. Of these, $6! = 720$ ways have to be ruled out because that nervous person does not want to be first. Hence the number of ways for the seven people to greet the President is $5040 - 720 = \mathbf{4320}$.

4. (5 points) An unbiased coin is tossed 5 times. What is the expected value of the number of times you will see an H immediately followed by a T?

Let X be the random variable which counts the number of times an H is followed by a T in a sequence of 5 tosses. We seek to compute $E[X]$. Write $X = X_1 + X_2 + X_3 + X_4$, where $X_1 = 1$ if you see an H on the 1st toss and T on the 2nd, and $X_1 = 0$ otherwise. Similarly $X_2 = 1$ if H occurs on the 2nd toss and T on the 3rd, and $X_2 = 0$ otherwise. And so on. The random variables X_1, X_2, X_3, X_4 take only two values each, either 0 or 1. The probability of getting a 1 is $\frac{1}{4}$ since 1 is attained only for HT, while 0 is attained for HH, TH, TT. Thus $E[X_i] = \frac{1}{4} \times 1 + \frac{3}{4} \times 0 = \frac{1}{4}$ for $i = 1, 2, 3, 4$. By linearity of expectation, $E[X] = E[X_1] + E[X_2] + E[X_3] + E[X_4] = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \mathbf{1}$.

Another solution is brute-force: simply display the graph of the random variable X:

(HHHHH, 0), (HHHHT, 1), (HHHTH, 1), (HHHTT, 1), (HHTHH, 1), (HHTHT, 2), (HHTTH, 1), (HHTTT, 1), (HTHHH, 1), (HTHHT, 2), (HTHTH, 2), (HTHTT, 2), (HTTHH, 1), (HTTHT, 2), (HTTTH, 1), (HTTTT, 1), (THHHH, 0), (THHHT, 1), (THHTH, 1), (THHTT, 1), (THTHH, 1), (THTHT, 2), (THTTH, 1), (THTTT, 1), (TTHHH, 0), (TTHHT, 1), (TTHTH, 1), (TTHTT, 1), (TTTHH, 0), (TTTHT, 1), (TTTTH, 0), (TTTTT, 0),

Adding up the values $X(\omega)$, we obtain 32. Dividing by $|\Omega| = 32$, we get $E[X] = 1$.

5. (5 points) How many ways there are to divide 5 people into 2 or 3 groups?

(The order of the groups and the order of the people in each group do not matter. In your answer, you must state the actual number, which is a positive integer.)

The number of ways for n people to form k groups is the Stirling number $S(n, k)$. Hence our task is to compute $S(5, 2) + S(5, 3)$. In Stirling's triangle the rows are $1, 1$ for $n = 2$, $1, 3, 1$ for $n = 3$, $1, 7, 6, 1$ for $n = 4$, and $1, 15, 25, 10, 1$ for $n = 5$. Hence $S(5, 2) = 15$ and $S(5, 3) = 25$. The sum of these two Stirling numbers is **40**.

6. (5 points) Box A contains 1 white ball and 4 black balls. Box B contains 1 white ball and 3 black balls. We flip a fair coin. If the outcome is heads, then a ball from box A is selected at random. If the outcome is tails, then a ball from box B is selected at random. Suppose that a white ball is selected. What is the probability that the coin landed tails?

Let W be the event "a white ball is selected" and T be the event "the coin landed tails." The problem wants $P(T|W)$. By Bayes's theorem, that probability equals

$$\frac{P(W|T)P(T)}{P(W|T)P(T)+P(W|T^c)P(T^c)} = \frac{\frac{1}{4}\cdot\frac{1}{2}}{\frac{1}{4}\cdot\frac{1}{2}+\frac{1}{5}\cdot\frac{1}{2}} = \frac{\frac{1}{4}}{\frac{1}{4}+\frac{1}{5}} = \mathbf{\frac{5}{9}} = 0.55555...$$

7. (5 points) Give the definition of "random variable" and "event." To receive full credit you must define other words from probability that appear in your definitions.

A probability space is a set Ω, called the outcome space or sample space of the probability space, and a probability function $P : \{\text{all subsets of } \Omega\} \to [0,1]$ that satisfies three additional properties. These three properties are $P(\varnothing) = 0$, $P(\Omega) = 1$, and $P(A \cup B) = P(A) + P(B)$ for any two subsets A and B of Ω with $A \cap B = \varnothing$.

A random variable is a function $X \colon \Omega \to \mathbb{R}$; it is a rule that takes every outcome in the outcome space of some probability space and assigns to it a real number. An event is a subset of Ω; in other words, it is a group of possible outcomes in the sample space of some probability space. If there are n outcomes then there are 2^n events.

8. (5 points) Now let X be the most number of heads you get in a row after flipping a fair coin five times. For example, if you got "HHHTH" then $X = 3$; if you got "HTTHT" then $X = 1$; and if you got "TTTTT" then $X = 0$.

(a) Explain why X is a random variable by showing it fits your definition in 7.

For the given X, the outcome space of the relevant probability space is:

$$\Omega = \{\text{all 5-letter strings of Hs and Ts}\},$$

so Ω has $2^5 = 32$ elements. The associated probability function is

$$P(A) = \frac{\text{number of elements in } A}{32} \qquad \text{for any subset } A \text{ of } \Omega.$$

Notice that X is a function from Ω to the real numbers; it is a rule that takes any element of Ω, i.e. any five-character string of Hs and Ts, and returns a real number.

(b) Explain why "$X \geq 2$" is an event by showing it fits your definition in 7.

The expression "$X \geq 2$" is short for "the set of all outcomes ω in Ω for which $X(\omega) \geq 2$", which is a subset of Ω. For example, you would include HHTTT, THHTH, and HHHTH as part of "$X \geq 2$".