

# UC Berkeley Math 10B, Spring 2015: Final Exam

Prof. Sturmfels, May 14, SOLUTIONS

1. (8 points)

- (a) What does the Central Limit Theorem say? State it in your own words.
- (b) A coin is tossed 100 times, resulting in 30 heads and 70 tails. Find a 95% confidence interval for the true probability  $p$  of getting heads.
- (c) Explain in what sense your interval has a 95% chance of containing  $p$ .

(a) Suppose that  $X_1, X_2, X_3, \dots$  are independent and identically distributed discrete random variables, with common expected value  $\nu$  and standard error  $\sigma$ . Then

$$Z = \frac{(X_1 + \dots + X_n) - n\nu}{\sigma\sqrt{n}}$$

is also a random variable. The Central Limit Theorem states that, for  $n \rightarrow \infty$ , the distribution of the random variable  $Z$  converges to the standard normal distribution, with mean 0 and variance 1.

(b) A 95% confidence interval for the parameter in  $n$  Bernoulli trials is

$$\left( \hat{p} - 2\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}, \hat{p} + 2\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \right).$$

In our situation, we have  $\hat{p} = 3/10$  and  $n = 100$ , so the interval is

$$\left( \frac{3}{10} - 2\frac{\sqrt{(3/10) \cdot (7/10)}}{10}, \frac{3}{10} + 2\frac{\sqrt{(3/10) \cdot (7/10)}}{10} \right) = \frac{1}{10} \left( 3 - \frac{\sqrt{21}}{5}, 3 + \frac{\sqrt{21}}{5} \right).$$

(c) For the normal distribution, 95% of the area under the curve is within two standard deviations of the mean. If we are using a biased coin with  $p = 3/10$ , and we toss it many many times, then, by the Central Limit Theorem, the probability is 95% that the sample mean will be in the corresponding interval around  $p$ . In our statistical analysis, we do not know the true parameter  $p$ , so we are using the sample mean  $\hat{p}$  instead. We are 95% confident that  $p$  lies within this interval around  $\hat{p}$ .

2. (8 points) Find all eigenvalues and all eigenvectors of the matrix  $A$ , where:

$$A = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}.$$

The characteristic polynomial of  $A$  is:

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} -2 - \lambda & 1 & 1 \\ 1 & -2 - \lambda & 1 \\ 1 & 1 & -2 - \lambda \end{vmatrix} \\ &= (-2 - \lambda) \begin{vmatrix} -2 - \lambda & 1 \\ 1 & -2 - \lambda \end{vmatrix} - 1 \begin{vmatrix} 1 & 1 \\ 1 & -2 - \lambda \end{vmatrix} + 1 \begin{vmatrix} 1 & -2 - \lambda \\ 1 & 1 \end{vmatrix} \\ &= -(\lambda + 2)((-2 - \lambda)^2 - 1^2) - (1(-2 - \lambda) - 1 \cdot 1) + (1 \cdot 1 - 1(-2 - \lambda)) \\ &= -(\lambda + 2)(\lambda^2 + 4\lambda + 3) + (\lambda + 3) + (\lambda + 3) \\ &= (\lambda + 3)(-(\lambda + 2)(\lambda + 1) + 1 + 1) \\ &= (\lambda + 3)(-\lambda^2 - 3\lambda) = -\lambda(\lambda + 3)^2. \end{aligned}$$

The eigenvalues of  $A$  are the roots of this polynomial, so they are  $\boxed{0 \text{ and } -3}$ .

To find the eigenvectors corresponding to  $\lambda = -3$ , we solve  $(A - (-3)I)\mathbf{x} = \mathbf{0}$  for  $\mathbf{x}$ :

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix} \xrightarrow[\substack{R_2 \rightarrow R_2 - R_1 \\ R_3 \rightarrow R_3 - R_1}]{R_2 \rightarrow R_2 - R_1} \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

So  $x_1 + x_2 + x_3 = 0$ , which means  $x_1 = -x_2 - x_3$ . So, the eigenvectors for  $\lambda = -3$

are  $\boxed{\mathbf{x} = \begin{pmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{pmatrix}}$  for any real numbers  $x_2$  and  $x_3$  that are not both 0.

For  $\lambda = 0$ , we solve  $(A - 0I)\mathbf{x} = \mathbf{0}$  for  $\mathbf{x}$ :

$$\begin{aligned} \begin{pmatrix} -2 & 1 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{pmatrix} &\xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 1 & -2 & 1 & 0 \\ -2 & 1 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{pmatrix} \xrightarrow{\substack{R_2 \rightarrow R_2 + 2R_1 \\ R_3 \rightarrow R_3 - R_1}} \begin{pmatrix} 1 & -2 & 1 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 3 & -3 & 0 \end{pmatrix} \\ &\xrightarrow{R_2 \rightarrow -\frac{1}{3}R_2} \begin{pmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 3 & -3 & 0 \end{pmatrix} \xrightarrow{\substack{R_1 \rightarrow R_1 + 2R_2 \\ R_3 \rightarrow R_3 - 3R_2}} \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

So  $x_1 - x_3 = 0$  and  $x_2 - x_3 = 0$ , i.e.  $x_1 = x_3$  and  $x_2 = x_3$ . We conclude that

$\mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$  and its nonzero constant multiples are the eigenvectors for  $\lambda = 0$ .

3. (8 points) A quiz question asks:

*How many seven-digit bitstrings have at least three 1s?*

A student gives the following solution:

You have to choose which three of the seven digits are 1s, and there are  $\binom{7}{3}$  ways to do that. Once you choose those three digits, each of the other four digits can be either 0 or 1, so overall there are  $\binom{7}{3} \cdot 2^4 = 35 \cdot 16 = \boxed{560}$  such bitstrings.

- (a) Compute the total number of seven-digit bitstrings. Fully simplify your answer.
- (b) Using your answer to (a), explain why the student's answer cannot be correct.
- (c) What is the correct answer to the quiz question? Fully simplify your answer.
- (d) Where is the mistake in the student's solution? Explain carefully.

**(a)** Since there are two choices for each of the seven digits, the total number of seven-digit bitstrings is  $2^7 = 2^4 \cdot 2^3 = 16 \cdot 8 = \boxed{128}$ .

**(b)** The number of seven-digit bitstrings with at least three 1s can't be bigger than the total number of seven-digit bitstrings. The answer to the quiz must be  $\leq 128$ .

**(c)** To find the number of seven-digit bitstrings with at least three 1s, subtract the number of bitstrings with zero, one, or two 1s from the total number of bitstrings. There are  $\binom{7}{0}$  bitstrings with zero 1s,  $\binom{7}{1}$  with one 1, and  $\binom{7}{2}$  with two 1s. [Choose which zero, one, or two of the seven digits are 1s.] That means the answer is

$$128 - \binom{7}{0} - \binom{7}{1} - \binom{7}{2} = 128 - 1 - 7 - \frac{7 \cdot 6}{2} = 120 - 21 = \boxed{99}.$$

**(d)** The student's method counts strings with four or more 1s multiple times, so the student's answer is bigger than the correct answer. If a string has  $n$  1s, where  $3 \leq n \leq 7$ , then the student's method will count that string  $\binom{n}{3}$  times, one for each way of picking three 1s from the  $n$ , instead of just once. Indeed,  $\sum_{n=3}^7 \binom{7}{n} \binom{n}{3} = 560$ .

4. (8 points) Determine all pairs of functions  $y_1(t), y_2(t)$  which satisfy:

$$\begin{aligned}y_1'(t) &= -2y_1(t) + 4y_2(t), & y_1(0) &= 1, \\y_2'(t) &= -2y_1(t) + 2y_2(t), & y_2(0) &= 2.\end{aligned}$$

Your final answer should not contain the complex number  $i = \sqrt{-1}$ .

The two eigenvalues of the matrix

$$A = \begin{pmatrix} -2 & 4 \\ -2 & 2 \end{pmatrix}$$

are  $2i$  and  $-2i$ , because the characteristic polynomial is

$$(-2 - \lambda) \cdot (2 - \lambda) - 4 \cdot (-2) = \lambda^2 + 4 = (\lambda - 2i)(\lambda + 2i).$$

An eigenvector for  $2i$  is  $\begin{pmatrix} 1 - i \\ 1 \end{pmatrix}$ , and an eigenvector for  $-2i$  is  $\begin{pmatrix} 1 + i \\ 1 \end{pmatrix}$ .

Let  $\mathbf{y}(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$ . Our aim is to solve the system  $\mathbf{y}'(t) = A\mathbf{y}(t)$ ,  $\mathbf{y}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ .

From the eigenvectors and eigenvalues, the general complex solution to  $\mathbf{y}' = A\mathbf{y}$  is:

$$\mathbf{y}(t) = C_1 e^{2it} \begin{pmatrix} 1 - i \\ 1 \end{pmatrix} + C_2 e^{-2it} \begin{pmatrix} 1 + i \\ 1 \end{pmatrix}.$$

To get real solutions, we find the real and imaginary parts of  $e^{2it} \begin{pmatrix} 1 - i \\ 1 \end{pmatrix}$ . We have:

$$\begin{aligned}e^{2it} \begin{pmatrix} 1 - i \\ 1 \end{pmatrix} &= (\cos 2t + i \sin 2t) \begin{pmatrix} 1 - i \\ 1 \end{pmatrix} \\&= \begin{pmatrix} (\cos 2t + i \sin 2t) - (i \cos 2t - \sin 2t) \\ \cos 2t + i \sin 2t \end{pmatrix} \\&= \begin{pmatrix} (\cos 2t + \sin 2t) + i(\sin 2t - \cos 2t) \\ \cos 2t + i \sin 2t \end{pmatrix} \\&= \begin{pmatrix} \cos 2t + \sin 2t \\ \cos 2t \end{pmatrix} + i \begin{pmatrix} \sin 2t - \cos 2t \\ \sin 2t \end{pmatrix}.\end{aligned}$$

So the real and imaginary parts of  $e^{2it} \begin{pmatrix} 1 - i \\ 1 \end{pmatrix}$  are  $\begin{pmatrix} \cos 2t + \sin 2t \\ \cos 2t \end{pmatrix}$  and  $\begin{pmatrix} \sin 2t - \cos 2t \\ \sin 2t \end{pmatrix}$ .

Then  $\begin{pmatrix} \cos 2t + \sin 2t \\ \cos 2t \end{pmatrix}$  and  $\begin{pmatrix} \sin 2t - \cos 2t \\ \sin 2t \end{pmatrix}$  are two real solutions to  $\mathbf{y}'(t) = A\mathbf{y}(t)$  that are not constant multiples of each other. Then the general real solution is:

$$\mathbf{y}(t) = C_1 \begin{pmatrix} \cos 2t + \sin 2t \\ \cos 2t \end{pmatrix} + C_2 \begin{pmatrix} \sin 2t - \cos 2t \\ \sin 2t \end{pmatrix}$$

for all real numbers  $C_1$  and  $C_2$ .

To work in the initial conditions, we put  $t = 0$  in the general solutions. This gives

$$C_1 \begin{pmatrix} 1 + 0 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} 0 - 1 \\ 0 \end{pmatrix} = \begin{pmatrix} C_1 - C_2 \\ C_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

We conclude that  $C_1 = 2$ , and  $C_2 = C_1 - 1 = 1$ . The solution to our ODE system is

$$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \mathbf{y}(t) = 2 \begin{pmatrix} \cos 2t + \sin 2t \\ \cos 2t \end{pmatrix} + 1 \begin{pmatrix} \sin 2t - \cos 2t \\ \sin 2t \end{pmatrix} = \begin{pmatrix} \cos 2t + 3 \sin 2t \\ 2 \cos 2t + \sin 2t \end{pmatrix}.$$

5. (8 points)

(a) Find a solution to the initial value problem

$$y' = t\sqrt{y}, \quad y(0) = 0.$$

(b) Find a solution to the initial value problem

$$y' = t\sqrt{y}, \quad y(0) = 1.$$

(c) Does the following initial value problem have a solution and, if yes, is it unique?

$$y' = 2\sqrt{y}, \quad y(0) = 0.$$

(a) The constant function  $y \equiv 0$  satisfies both the differential equation  $y' = t\sqrt{y}$  and the initial condition  $y(0) = 0$ , so  $y \equiv 0$  solves this initial value problem. Another solution is

$$y = \frac{1}{16}t^4.$$

(b) Since  $y$  is not zero, we can divide by  $y$  and consider the equation

$$\frac{y'}{\sqrt{y}} = t.$$

This yields  $\int y^{-1/2} dy = \int t dt$  and hence  $2y^{1/2} = \frac{1}{2}t^2 + C$  for some constant  $C$ . So,

$$y = \left( \frac{t^2/2 + C}{2} \right)^2.$$

The initial condition  $y(0) = 1$  implies  $C = 2$ , and hence our solution is

$$y(t) = \left( \frac{t^2}{4} + 1 \right)^2$$

(c) Yes, this initial value problem has a solution, namely the constant function  $y \equiv 0$ . However, the solution is not unique because  $y = t^2$  is also a solution.

6. (8 points) Let  $I_2$  denote the  $2 \times 2$ -identity matrix. For each part, check that the example you give really is an example.

- (a) Give two  $2 \times 2$ -matrices  $A$  and  $B$  such that  $AB \neq BA$ .
- (b) Give a  $2 \times 2$ -matrix  $A$  such that  $A \neq I_2$ ,  $A \neq -I_2$ , and  $AA = I_2$ .
- (c) Give a  $2 \times 2$ -matrix  $A$  such that  $A \neq I_2$ ,  $A \neq -I_2$ , and  $A^T A = I_2$ .
- (d) Give two  $2 \times 3$ -matrices  $A$  and  $B$  such that  $AB^T = I_2$ .

(a) Almost all pairs of matrices have this property. We can take  $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$  and  $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ . Then  $AB = B$  but  $BA$  is the zero matrix.

(b) There are infinitely many  $2 \times 2$ -matrices whose square is the identity matrix. For instance, for any non-zero real number  $x$  you can take  $A = \begin{pmatrix} 2 & x \\ -3/x & -2 \end{pmatrix}$ . To check, we multiply:

$$A \cdot A = \begin{bmatrix} 2 & x \\ -3/x & -2 \end{bmatrix} \begin{bmatrix} 2 & x \\ -3/x & -2 \end{bmatrix} = \begin{bmatrix} 2 \cdot 2 - x \cdot 3/x & 2 \cdot x - x \cdot 2 \\ -6/x + 6/x & -3/x \cdot x + 4 \end{bmatrix} = I_2.$$

(c) There are infinitely many  $2 \times 2$ -matrices whose inverse equals its transpose. For instance, every rotation matrix  $A = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$  has this property.

(d) A simple solution is  $A = B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ . But there are infinitely many others.



7. (8 points) You want to put 9 indistinguishable apples into 4 distinguishable boxes, where some of the boxes may be empty.

(a) How many ways can you do this? Simplify your answer completely.

(b) A raccoon finds the box with the most apples and eats all of the apples in it.

[If there is a tie, the raccoon picks one of the boxes with the most apples.] What is the minimum number of apples that the raccoon is guaranteed to eat? Explain carefully.

(a) The number of ways of putting  $b$  indistinguishable balls [apples] into  $u$  distinguishable boxes is  $\binom{b+u-1}{b} = \binom{b+u-1}{u-1}$ . For this problem,  $b = 9$  and  $u = 4$ , so the answer is:

$$\binom{12}{9} = \binom{12}{3} = \frac{12 \cdot 11 \cdot 10}{3 \cdot 2 \cdot 1} = \frac{12 \cdot 11 \cdot 10}{6} = 2 \cdot 11 \cdot 10 = 22 \cdot 20 = \boxed{220}.$$

(b) By the Pigeonhole Principle, if you put 9 pigeons [apples] into 4 holes [boxes], then at least one hole will have at least  $\lceil \frac{9}{4} \rceil = \lceil 2.25 \rceil = 3$  pigeons. So at least one box will have at least 3 apples. Therefore, the raccoon will eat at least 3 apples.

We also need to check that it isn't always possible for the raccoon to eat 4 or more apples; if it were, then the minimum would be 4 or more. But if you put 3 apples in one box and 2 in each of the other three boxes, then the raccoon will eat 3.

So  $\boxed{3}$  is the minimum.

8. (8 points)

- (a) State the mathematical problem that is solved by the *Least Squares Method*.  
(b) Find the least square line  $y = \beta_0 + \beta_1 x$  for the data points given by the table

$x$	0	1	2	3	4
$y$	6	3	1	0	0

- (c) Estimate the value of  $y$  when  $x = 7/2$ .

**(a)** Given  $n$  data points  $(x_1, y_1), \dots, (x_n, y_n)$  in the plane, the Least Squares Method finds the unique pair of real numbers  $\beta_0$  and  $\beta_1$  that minimize the least squares error

$$\sum_{i=1}^n (y_i - (\beta_0 + \beta_1 x_i))^2.$$

The resulting straight line  $y = \beta_0 + \beta_1 x$  best approximates the given data points.

**(b)** The formulas for the numbers  $\beta_1$  and  $\beta_0$  in the line-of-best-fit through  $n$  points  $(x_1, y_1), \dots, (x_n, y_n)$  are:

$$\beta_1 = \frac{n(\sum_{i=1}^n x_i y_i) - (\sum_{i=1}^n x_i)(\sum_{i=1}^n y_i)}{n(\sum_{i=1}^n (x_i)^2) - (\sum_{i=1}^n x_i)^2}, \quad \beta_0 = \bar{y} - \beta_1 \bar{x},$$

where  $\bar{x}$  is the average value of the  $x_i$ s, i.e.  $\frac{\sum_{i=1}^n x_i}{n}$  and similarly  $\bar{y} = \frac{\sum_{i=1}^n y_i}{n}$ .

To find the least-squares line for the  $n = 5$  data points, we fill in the following table:

	$x_i$	$y_i$	$(x_i)^2$	$x_i y_i$
$i = 1$	0	6	0	0
$i = 2$	1	3	1	3
$i = 3$	2	1	4	2
$i = 4$	3	0	9	0
$i = 5$	4	0	16	0
Sum	10	10	30	5

We therefore find that:

$$\beta_1 = \frac{5 \cdot 5 - 10 \cdot 10}{5 \cdot 30 - 10^2} = \frac{25 - 100}{150 - 100} = -\frac{75}{50} = -\frac{3}{2}$$
$$\beta_0 = 2 - ((-3/2) \cdot 2) = 5.$$

So the least-squares line is  $y = 5 - \frac{3}{2}x$ .

(c) To find an estimate for  $y$  when  $x = 7/2 = 3.5$ , we plug in  $\frac{7}{2}$  into the formula for the least-squares line. This gives

$$y \approx 5 - \frac{3}{2} \cdot \frac{7}{2} = \frac{20}{4} - \frac{21}{4} = \boxed{-\frac{1}{4} = -0.25}.$$

9. (8 points)

We define two random variables  $X$  and  $Y$  as follows. You pick a number  $X$  with equal probability from the set  $\{1, 2, 3\}$ , and you set

$$Y = \begin{cases} 1 & \text{if } X \text{ is even,} \\ 0 & \text{if } X \text{ is odd.} \end{cases}$$

(a) Show that the covariance  $\text{Cov}[X, Y]$  is zero.

**Reminder:** For any random variables  $X$  and  $Y$ ,  $\text{Cov}[X, Y] = E[XY] - E[X]E[Y]$ .

(b) Are  $X$  and  $Y$  independent? Why or why not? Explain carefully.

**(a)** The random variable  $X$  takes the values 1, 2 or 3 each with equal probability  $1/3$ . The expected value is  $E[X] = \frac{1}{3}(1+2+3) = 2$ . The random variable  $Y$  outputs 1 with probability  $1/3$  and 0 with probability  $2/3$ , so  $E[Y] = 1 \cdot \frac{1}{3} + 0 \cdot \frac{2}{3} = 1/3$ .

The random variable  $XY$  multiplies the output of  $X$  with the output of  $Y$ . The possible values are  $2 = 2 \cdot 1$ , with probability  $1/3$ , and  $0 = 1 \cdot 0 = 3 \cdot 0$ , with probability  $2/3$ . That means  $E[XY] = 2 \cdot \frac{1}{3} + 0 \cdot \frac{2}{3} = 1/3$ .

We conclude that  $\text{Cov}[X, Y] = E[XY] - E[X]E[Y] = \frac{1}{3} - 2 \cdot \frac{1}{3} = 0$ , as needed.

**(b)** If  $X$  and  $Y$  are independent then  $\text{Cov}[X, Y] = 0$ , but the reverse isn't always true. To check that a discrete random variable  $X$  with outputs  $a_1, \dots, a_m$  and a discrete random variable  $Y$  with outputs  $b_1, \dots, b_n$  are independent, we must check that, for each possible output  $a_i$  of  $X$  and each possible output  $b_j$  of  $Y$ , the events “ $X$  outputs  $a_i$ ” and “ $Y$  outputs  $b_j$ ” are independent, i.e. we must check:

$$P(X = a_i \text{ and } Y = b_j) = P(X = a_i) \cdot P(Y = b_j) \quad \text{for all } a_i \text{ and } b_j$$

If the previous equation holds for all  $a_i$  and  $b_j$ , then  $X$  and  $Y$  are independent. If the previous equation doesn't work for even one  $a_i$  and  $b_j$ , then they are not independent. For our problem,  $X$  can output 1, 2, or 3, and  $Y$  can output 0 or a 1. So, to check if  $X$  and  $Y$  are independent, we need to check the following six equations:

$$\begin{array}{ll} P(X = 1 \text{ and } Y = 0) = P(X = 1) \cdot P(Y = 0), & P(X = 1 \text{ and } Y = 1) = P(X = 1) \cdot P(Y = 1), \\ P(X = 2 \text{ and } Y = 0) = P(X = 2) \cdot P(Y = 0), & P(X = 2 \text{ and } Y = 1) = P(X = 2) \cdot P(Y = 1), \\ P(X = 3 \text{ and } Y = 0) = P(X = 3) \cdot P(Y = 0), & P(X = 3 \text{ and } Y = 1) = P(X = 3) \cdot P(Y = 1). \end{array}$$

If all six equations are correct, then  $X$  and  $Y$  are independent. If even one equation is incorrect, then we don't need to check any of the other 5 equations; we know that  $X$  and  $Y$  are not independent. Let's consider the equation

$$P(X = 1 \text{ and } Y = 1) = P(X = 1) \cdot P(Y = 1).$$

We have  $P(X = 1 \text{ and } Y = 1) = 0$  because there is no way that  $X$  and  $Y$  can both be 1; if  $X = 1$  then  $Y = 0$  because 1 is odd. However,  $P(X = 1) = \frac{1}{3}$  and  $P(Y = 1) = \frac{1}{3}$ , and  $0 \neq \frac{1}{3} \cdot \frac{1}{3}$ . Therefore,  $X$  and  $Y$  are not independent.

In fact, none of the six equations are true, as the following table shows:

$a$	$b$	$P(X = a \text{ and } Y = b)$	$P(X = a) \cdot P(Y = B)$	Are the two equal?
1	0	$\frac{1}{3}$	$\frac{1}{3} \cdot \frac{2}{3} = \frac{2}{9}$	No
2	0	0	$\frac{1}{3} \cdot \frac{2}{3} = \frac{2}{9}$	No
3	0	$\frac{1}{3}$	$\frac{1}{3} \cdot \frac{2}{3} = \frac{2}{9}$	No
1	1	0	$\frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}$	No
2	1	$\frac{1}{3}$	$\frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}$	No
3	1	0	$\frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}$	No

10. (8 points) You put a cup of coffee initially at  $85^{\circ}\text{C}$  into a refrigerator, which stays at  $5^{\circ}\text{C}$ . After 15 minutes, your coffee has cooled to  $65^{\circ}\text{C}$ . Find a formula for the temperature in Celsius of the coffee  $t$  minutes after you put it in the refrigerator.

**Reminder:** Newton's law of cooling states the rate of change of the temperature of an object is proportional to the difference between the temperature of the object and the temperature of its surroundings.

Let  $y(t)$  be the temperature (in  $^{\circ}\text{C}$ ) of the coffee after  $t$  minutes. We are told that

$$y(0) = 85 \quad \text{and} \quad y(15) = 65.$$

Newton's Law tells us that, for some constant  $k$ , we have

$$y'(t) = k(y(t) - 5).$$

This is a separable differential equation. Since  $y(t)$  is not a constant function, we can divide by  $y - 5$  and write the differential equation as

$$\frac{dy}{y - 5} = k \cdot dt.$$

We find  $\ln|y - 5| = kt + C$ , and hence

$$y(t) = C' \cdot \exp(kt) + 5$$

Here  $C$  and  $C'$  are suitable constants. From  $y(0) = 85$ , we find  $C' = 80$ . The condition  $y(15) = 65$  gives  $80 \cdot \exp(15k) + 5 = 65$ , hence  $\exp(15k) = 3/4$ , and therefore  $k = \frac{1}{15}\ln(3/4)$ . The temperature at time  $t$  is thus given by the function

$$y(t) = 80 \cdot \exp\left(\frac{1}{15}\ln(3/4) \cdot t\right) + 5.$$

We prefer to write our solution as

$$y(t) = 80 \cdot (3/4)^{t/15} + 5.$$