1. (20 points, 10 points each) Complete the following definitions. You may use, without defining them, any terms or symbols that our text defines before defining the word or symbol asked for. Your definitions do not have to have exactly the same wording as those in the text, but for full credit they should be clear, and mean the same thing as those definitions.

(a) If H and K are groups, and φ is *Answer: a homomorphism from* K *to* $\mathrm{Aut}(H)$, then by the semidirect product $H \rtimes_\varphi K$ one means the group whose elements are ordered pairs (h, k) with $h \in H$ and $k \in K$, and whose group operation is given by the law $(h_1, k_1)(h_2, k_2) =$

Answer: $(h_1 \varphi(k_1)(h_2),\ k_1 k_2)$.

(b) (10 points) If R is a ring, then a left ideal I of R means

Answer: a subring $I \subseteq R$ *such that for all* $r \in R$, *we have* $rI \subseteq I$.

2. (40 points, 10 points each.) For each of the items listed below, either *give an example* with the property stated, or give a brief reason why *no such example exists.*

If you give an example, you do *not* have to prove that it has the property stated; however, your examples should be specific; i.e., even if there are many objects of a given sort, you should name a particular one. If you give a reason why no example exists, don't worry about giving reasons for your reasons; a simple statement will suffice.

(a) A nonabelian simple group. *Answer:* A_5

(b) A nonabelian group of order 33.

Answer: Does not exist: $|\mathrm{Aut}(Z_{11})| = 10$ *is not divisible by* 3.

(c) Two nonisomorphic, noncyclic, abelian groups of order 81 ($= 3^4$).

Answer: Any two of $Z_{27} \times Z_3$, $Z_9 \times Z_9$, $Z_9 \times Z_3 \times Z_3$, $Z_3 \times Z_3 \times Z_3 \times Z_3$.

(d) A ring R, a subring $S \subseteq R$, and an element $u \in S$ which is a unit of R but not a unit of S.

Answer: $R = \mathbb{Q}$, $S = \mathbb{Z}$, $u = 2$.

3. (20 points) Let G be a finite group which acts on a set A, let p be a prime, and let P be a Sylow p-subgroup of G. We shall say that P *acts on* A *without fixed points* if for every nonidentity element $x \in P$ and every $a \in A$, we have $xa \neq a$.

Show that if P acts on A without fixed points, then *every* Sylow p-subgroup Q of G acts on A without fixed points. You may use any results proved in our readings.

Answer: Let Q *be any Sylow p-subgroup of* G. *By a result in the readings,* Q *is a conjugate of* P, *say* $Q = gPg^{-1}$. *Now suppose some* $a \in A$ *is fixed by some element of* Q. *Writing that element of* Q *as* ghg^{-1} *where* $h \in P$, *we have* $ghg^{-1}a = a$. *Applying* g^{-1} *to both sides of this equation, we get* $hg^{-1}a = g^{-1}a$; *so* $g^{-1}a$ *is fixed by* $h \in P$. *By assumption, this can only happen if* $h = 1$; *so our given element of* Q *is* $g1g^{-1} = 1$, *as required to show that* Q *acts without fixed points.*

(*This proof can also be formulated in terms of the result* $G_{ga} = g\,G_a\,g^{-1}$.)

4. (20 points) Suppose R is a commutative ring, and e is an element of R which satisfies $e^2 = e$. (Such an element is called an *idempotent*.) Show that the map $\varphi: R \to R$ given by $\varphi(r) = er$ is a ring homomorphism, and that $\ker(\varphi) = \{s - es \mid s \in R\}$.

Answer: To show that φ *is a ring homomorphism, we compute*

$$\varphi(r+s) = e(r+s) = er + es = \varphi(r) + \varphi(s).$$
$$\varphi(rs) = ers = e^2rs = eres = \varphi(r)\,\varphi(s).$$

Next, note that any element of the form $s - es$ *satisfies*

$$\varphi(s-es) = e(s-es) = es - e^2s = es - es = 0,$$

so such elements indeed belong to $\ker(\varphi)$. *Conversely, if* $r \in \ker(\varphi)$, *I claim that* $r = r - er$, *showing that* r *has the desired form. Indeed,*

$$r - er = r - \varphi(r) = r - 0 = r,$$

as required.

☛ Reminder: The reading for Friday, April 3, is #22 ☚