Math 185 Final Exam
May 23, 2003

NAME (printed) : ______________________________
(Family Name) (First Name)

Signature : ______________________________

Student Number : ______________________________

(1) Do not open this test booklet until told to do so

(2) Do all your work in this test booklet

(3) Show all your work

(4) Check that there are 10 problems

(5) No calculators

(6) DON'T PANIC

1	2	3	4	5	6	7	8	9	10

TOTAL

1 a: (3 pts) Let $C = \{e^{i\theta}, 0 \leq \theta \leq 2\pi\}$ be the contour around 0 of radius 1. Evaluate

$$\int_C e^{z^2}\, dz$$

b: (5 pts) Let $C = \{e^{i\theta}, 0 \leq \theta \leq 2\pi\}$ be the contour around 0 of radius 1. Evaluate

$$\int_C \frac{1}{z^2 + z - 1}\, dz$$

c: (4 pts) Find the Residue at $z = 0$ for

$$f(z) = \frac{\ln(1-z)}{z^{10}}$$

2 a: (5 pts) Suppose that $f(z)$ is entire and that the harmonic function $v(x, y) = \mathrm{Im}(f(z))$ is bounded. Show that $v(x, y)$ must be a constant.

b: (5 pts) Let $f(z)$ be analytic, non-constant and non-zero on $|z| \leq 1$. Show that $|f(z)|$ attains its minimum on the boundary $|z| = 1$.

3 a: (5 pts) Show that

$$\int_{-\infty}^{\infty} \frac{\sin(x)}{x}\, dx = \pi$$

b: (5 pts) Show that

$$\int_{-\infty}^{\infty} \frac{1}{x^2 + x + 1}\, dx = \frac{2\pi}{\sqrt{3}}$$

4 a: (4 pts) Find the first 4 terms of the Taylor series, centered at $z = 0$, for

$$f(z) = \frac{\sin(z)}{1 - z}$$

and give the radius of convergence.

b: (4 pts) Give the Laurent series for $\frac{1}{z - z^2}$, centered at $z = 0$, for $|z| > 1$.

c: (4 pts) Let f be an entire function. Let $f(z) = 0$ on $z = [-1, 1]$. Prove that $f(z) = 0$ for all complex number $z \in \mathbb{C}$.

5 a: (6 pts) Define, and given an example for each.

- $f(z)$ has a zero of order three at 2,

- $f(z)$ has a essential singularity at i

- $f(z)$ has a removable pole at 0.

b: (4 pts) Let f be an entire function with a zero of order 4 at $z = 0$ and no other zeros. Let C be any simple closed contour around 0. Evaluate

$$\int_C \frac{f'(z)}{f(z)}\, dz$$

6 a: (2 pts) State the Fundamental Theorem of Algebra.

b: (3 pts) How many roots does the function $f(z) = 4z^7 + 7z^4 + 1$ have within the circle $|z| = 1$?

c: (3 pts) Where is the function $f(z) = 4z^7 + 7z^4 + 1$ conformal?

7 a: (4 pts) Find the image of the circle $x^2 + y^2 + 2x - 4y - 9 = 0$ under the map $w = 1/z$.

b: (4 pts) Find a linear fractional transformation that takes 1 to i, takes ∞ to 1 and has a fixed point at 0.

8 a: (3 pts) Define what it means for a set D to be

- simply connected,

- multiply connected,

- bounded

b: (4 pts) Find the radius of convergence for the following Taylor series, centered at $z = 0$. Do *not* find the Taylor series!!

- $\frac{1}{\cos(z)+1}$

- $\frac{1}{z^2-z-1}$

c: (3 pts) Let $f(z) = (1+i)(x+y)$ (where $z = x + iy$). Show that this function is not differentiable anywhere.

9 a: (5 pts) Evaluate

$$\int_0^{2\pi} \frac{1}{2+\cos(\theta)}\, d\theta$$

b: (5 pts) Find the inverse Laplace transform of

$$F(s) = \frac{2s^3}{s^4 - 4}$$

10 a: (4 pts) Let $f(z)$ be an entire function such that $f(z) = f(z+1)$ and $f(z) = f(z+i)$ for all complex numbers z. Show that $f(z)$ is a constant.

b: (6 pts) Let $f(z)$ be an analytic function, except possibly for poles of finite order. Further let $f(z) = f(z+1)$ and $f(z) = f(z+i)$ for all complex numbers z. Let C be the square contour around the unit square $0 \leq x \leq 1$ and $0 \leq y \leq 1$. Further, let f have no poles or zeros on C. Show that

$$\Delta_C \arg f(z) = 0$$

(For partial credit **(4 pts)**, show instead that $\int_C f(z) = 0$.)