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Math 128A final exam 24 May 2003

1. (a) Find the Newton form of the quadratic polynomial interpolating $f(x) = \ln x$ at $x = 1, e$ and e^2.

(b) Show that the error is no larger than 512 for $1 \le x \le e^2$.

(c) Evaluate the actual error at $x = 5$.

2. Let

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}.$$

(a) Find a lower triangular matrix L and an upper triangular matrix U such that $PA = LU$ ~~for~~ some permutation matrix P. Explain your procedure.

(b) Is A diagonally dominant? Positive definite? Does either answer affect the procedure and result of (a)?

(c) Is A invertible? Why or why not?

3.(a) For the initial value problem

$$y' = f(t,y), \quad y(0) = y_0$$

define the local truncation error τ_n of a numerical method

$$u_{n+1} = u_n + h\,\Phi(t_n, h; u_n, \ldots, u_{n-k}; f).$$

(b) Show that for the trapezoidal rule

$$u_{n+1} = u_n + h\left[\tfrac{1}{2} f(t_{n+1}, u_{n+1}) + \tfrac{1}{2} f(t_n, u_n)\right]$$

the local truncation error is $O(h^2)$.

(c) How can you use the trapezoidal rule (with various step sizes h) to construct a numerical method with local truncation error $O(h^4)$?

4. Determine the limit and speed of convergence for the iteration

$$x_{n+1} = 2x_n - \pi x_n^2, \quad x_0 = 1/3.$$

Use analysis (not a calculator).

5. Suppose $u \in \mathbb{R}^n$ is a unit vector ($\|u\|_2 = 1$).

(a) Explain the geometric meaning of the matrices

$$Q = uu^T, \quad P = I - uu^T, \quad R = I - 2uu^T.$$

(b) Find a 3x3 <u>orthogonal</u> matrix Q such that

$$Q \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}.$$

6. (a) Find the weights w_j of a quadrature formula

$$\int_0^1 f(x)\,dx \approx w_0 f(0) + w_1 f(1) + w_2 f'(1)$$

which has as high precision as possible.

(b) Without any additional calculation, find an equally accurate formula of the form

$$\int_0^1 f(x)\,dx \approx w_0^* f(0) + w_1^* f(1) + w_2^* f'(0).$$

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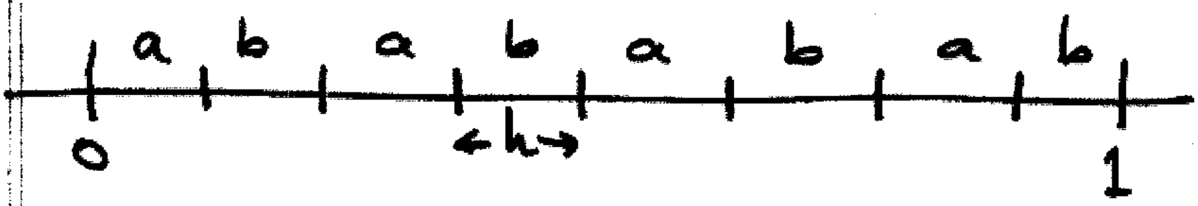
(c) Consider compounding the rules of (a) and (b) by applying them alternately on intervals of length $h = 1/2n$ from 0 to 1:



Write out the resulting rule in the form

$$\int_0^1 f(x)\,dx = a_0 f(0) + a_1 f(h) + a_2 f(2h) + \dots + a_{2n} f(2nh) + b_0 f'(0) + b_1 f'(h) + \dots + b_{2n} f'(2nh).$$

(d) Find the largest integer p such that the error of the rule in (c) is

$$O(h^p),$$

and determine by dimensional analysis the order q of differentiation that will make the error

$$O(f^{(q)} h^p).$$