MIDTERM FOR MATHEMATICS 195

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INSTRUCTIONS: This test is due in class on Wednesday, April 10. You may not discuss this test with any person, nor consult any book or article. You may look at your own notes, the handouts distributed in class, and also the old set of lecture notes, downloaded from my website at http: //www.math.berkeley.edu/~evans/.

Please come by and talk with me if you have any questions about these problems. Good luck.

PROBLEM #1. Let X and Y be *independent*, real–valued random variables defined on the same probability space. Assume that f_X is the density function for X, and f_Y is the density function for Y.

Show that the density function for $X + Y$ is given by the formula:

$$f_{X+Y}(z) = \int_{-\infty}^{\infty} f_X(z-y) f_Y(y)\, dy$$

for $z \in \mathbb{R}$.

(HINT: For each function $g : \mathbb{R} \to \mathbb{R}$, we have

$$E(g(X+Y)) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y) g(x+y)\, dx dy,$$

where $f_{X,Y}$ is the joint density function of X, Y. Use independence and let $z = x + y$.)

PROBLEM #2. Let $W(\cdot)$ be a one-dimensional Brownian motion. Show that

$$\lim_{m \to \infty} \frac{W(m)}{m} = 0 \quad \text{almost surely.}$$

(HINT: Fix $\epsilon > 0$ and define the event $A_m := \{|\frac{W(m)}{m}| \geq \epsilon\}$. Then $A_m = \{|X| \geq \sqrt{m}\epsilon\}$ for the $N(0,1)$ random variable $X = \frac{W(m)}{\sqrt{m}}$. Therefore

$$P(A_m) = \frac{2}{\sqrt{2\pi}} \int_{\sqrt{m}\epsilon}^{\infty} e^{-\frac{x^2}{2}}\, dx.$$

Apply the Borel–Cantelli Lemma.)

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PROBLEM #3. Let $f : [0,1] \to \mathbb{R}$ be continuous and define the *Bernstein polynomial*

$$b_n(x) := \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k}.$$

Prove that $b_n \to f$ uniformly on $[0,1]$ as $n \to \infty$, by providing the details for the following steps:

(i) Since f is uniformly continuous, for each $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that $|f(x) - f(y)| \le \epsilon$ if $|x - y| \le \delta(\epsilon)$.

(ii) Given $x \in [0,1]$, take a sequence of independent random variables X_k defined on some probability space $(\Omega, \mathcal{U}, P)$, such that $P(X_k = 1) = x, P(X_k = 0) = 1 - x$. Write $S_n = X_1 + \cdots + X_n$. Then $E(S_n) = nx$. Furthermore, $b_n(x) = E(f(\frac{S_n}{n}))$, since the probability that $S_n = k$ is $\binom{n}{k} x^k (1-x)^{n-k}$ for $k = 0, 1, \ldots, n$.

(iii) Therefore

$$|b_n(x) - f(x)| \le E(|f(\frac{S_n}{n}) - f(x)|) = \int_A |f(\frac{S_n}{n}) - f(x)| \, dP + \int_{A^c} |f(\frac{S_n}{n}) - f(x)| \, dP,$$

for $A = \{|\frac{S_n}{n} - x| \le \delta(\epsilon)\}$, $A^c = \{|\frac{S_n}{n} - x| > \delta(\epsilon)\}$

(iv) Then use Chebyshev's inequality to show that

$$|b_n(x) - f(x)| \le \epsilon + \frac{2M}{\delta(\epsilon)^2} V(\frac{S_n}{n}) = \epsilon + \frac{2M}{n\delta(\epsilon)^2} V(X_1),$$

for $M = \max |f|$. Conclude that $b_n \to f$ uniformly.