MATH 185 Spring 2001 Prof. Croot

Final Exam

1. Suppose $f(z)$ is entire, and satisfies the relations $f(z+1) = f(z)$, $f(z+i) = f(z)$. Show that $f(z)$ must be a constant.

2. Suppose $f(z)$ is meromorphic with only a simple pole at $z = i$ (analytic everywhere else). Show that there exists a number $z_0 \in \mathbf{R}$ such that $f(z_0) \notin \mathbf{R}$.

3. Use the $\epsilon - \delta$ definition of limits to show that

$$\lim_{z \to i} (\overline{z})^2 + 1 = 0.$$

4. Suppose that $T(z)$ is a linear fractional transformation, where the point mapped to infinity satisfies $|z_0| > 10$. Show that $T(z)$ maps both of the circles $|z| = 1$ and $|z-1| = 1$ to lines. State any and all properties of LFT's you use.

5. Evaluate

$$\int_0^\infty \frac{\cos x}{x^2+1} dx$$

using contour methods.

6. Find the Laurent expansions of

$$f(z) = \frac{z+3}{z^2(z^2+1)}$$

in $0 < |z| < 1$ and $|z| > 1$.

7. Suppose $f(z)$ is entire, $f(0) = 0$. Write $f(z) = u(x,y) + iv(x,y)$. Show that there exists a point (x_0, y_0) satisfying $x_0^2 + y_0^2 = 1$ where $u(x_0, y_0) = 0$.

8. Find a harmonic conjugate of

$$u(x,y) = x^3 - 3xy^2 + 3y^2 - 3x^2 + 2x - 7.$$

9. Evaluate

$$\int_C z^{1/3} dz,$$

where C is the semicircle $e^{i\theta}, 0 \le \theta \le \pi$, and where the branch of $\log z$ used, implicit in the $z^{1/3}$, has $3\pi/2 < \arg z < 7\pi/2$.

10. Suppose $f(z)$ is entire and satisfies

$$\frac{|f^{(n)}(0)|}{n!} > \sum_{\substack{j=0 \\ j \ne n}}^{\infty} \frac{|f^{(j)}(0)|}{j!}.$$

Prove that $f(z)$ has exactly n zeros (counting multiplicities) z_0 satisfying $|z_0| \le 1$.