MATH 126: MIDTERM 2, SPRING, 2000

Total score: 100 points.

Problem 1. *The left end of a bar of length l is kept at temperature 0, and its right end is kept at temperature U. Its temperature at $t = 0$ is $\phi(x) = 0$, $0 < x < l$. Find the temperature of the bar for $t > 0$ as follows.*

- (i) *(8 points) Find the steady state solution u_0 of the heat equation with these boundary conditions. That is, find a function $u_0 = u_0(x)$ on $[0, l]$ which satisfies $u_0''(x) = 0$ for $0 < x < l$, $u_0(0) = 0$, $u_0(l) = U$.*
- (ii) *(7 points) Let $v(x,t) = u(x,t) - u_0(x)$. Using that u solves $u_t = ku_{xx}$, show that v solves the same equation with homogeneous boundary conditions. What initial condition does v satisfy (i.e. what is $v(x,0)$)?*
- (iii) *(10 points) Separate variables in $v_t = kv_{xx}$, and show that the general solution with boundary conditions $v(0,t) = 0 = v(l,t)$ is*

$$v(x,t) = \sum_{n=1}^{\infty} B_n \sin(n\pi x/l) e^{-kn^2\pi^2 t/l^2}.$$

  You may use, without doing the calculation, that the eigenfunctions of $-d^2/dx^2$ on $[0,l]$, with homogeneous Dirichlet boundary conditions are (multiples of) $X_n(x) = \sin(n\pi x/l)$, $n \geq 1$ integer.
- (iv) *(10 points) Find the constants B_n.*
- (v) *(5 points) Find $\lim_{t\to\infty} u(x,t)$.*

Problem 2. *Let $A = -d^2/dx^2$ defined on C^2 functions f on $[0,l]$ which satisfy mixed boundary conditions: $f(0) = 0$, $f'(l) = 0$.*

- (i) *(10 points) Show that A is symmetric (with respect to the scalar product $(f,g) = \int_0^l f(x)\,\overline{g(x)}\,dx$), and that it is positive.*
- (ii) *(10 points) Find all eigenvalues and eigenfunctions of A.*
- (iii) *(10 points) Using the results of (i)-(ii), find*

$$\int_0^l \sin\left(\frac{(2n+1)\pi x}{2l}\right) \sin\left(\frac{(2m+1)\pi x}{2l}\right) dx$$

  for non-negative integers $n \neq m$.

Problem 3. *Consider the wave equation $u_{tt} = c^2 u_{xx}$ on the half-line $x > 0$ with inhomogeneous Dirichlet boundary condition $u(0,t) = h(t)$, h a given function. Suppose that $u(x,0) = 0$ and $u_t(x,0) = 0$ for $x > 0$.*

- (i) *(10 points) Show that $u(x,t) = 0$ if $t > 0$, $x > ct$. Sketch this region.*
- (ii) *(12 points) Find the solution u for all $x > 0$, $t > 0$.*
- (iii) *(8 points) Suppose that h is 0 near $t = 0$, and is C^∞ except at a point $t_0 > 0$. Where can you say that u is C^∞? Sketch the region.*

You may use in any part of the problem that if v solves $v_{tt} - c^2 v_{xx} = f$ on Δ, the backward characteristic triangle from (x,t), then

$$v(x,t) = \frac{v(x-ct,0) + v(x+ct,0)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} v_t(x',0)\,dx' + \frac{1}{2c}\int_{\Delta} f.$$