MATH 126: MIDTERM 1, SPRING, 2000

Total score: 100 points.

Problem 1.

(i) *(15 points) Find the general solution of the PDE*

$$u_x + xu_y = x^2$$

(ii) *(10 points) Now impose in addition the initial condition $u(0, y) = y$. Find u explicitly.*

Problem 2. *Suppose that $u = u(x,t)$ solves the heat equation*

$$u_t = ku_{xx},\ (x,t) \in (0,l) \times (0,\infty),\ k > 0,$$

with Dirichlet boundary conditions

$$u(0,t) = 0,\ u(l,t) = 0,$$

and initial condition $u(x,0) = \phi(x)$ where ϕ is a non-negative function.

(i) *(8 points) Show that $u(x,t) \geq 0$ for all $(x,t) \in (0,l) \times (0,\infty)$.*

(ii) *(8 points) Show that $u_x(0,t) \geq 0$, $u_x(l,t) \leq 0$ for $t > 0$. (Hint: consider difference quotients.)*

(iii) *(9 points) Let $Q(t) = \int_0^l u(x,t)\,dx$ be the total heat at time t. Show that Q is a decreasing function of time. (Hint: what is $Q'(t)$?)*

Problem 3. *Consider the PDE*

$$u_{tt} + u_{xt} - 2u_{xx} = 0, \quad (x,t) \in \mathbb{R} \times \mathbb{R}.$$

(i) *(8 points) What is its type?*

(ii) *(15 points) Find the general solution of the PDE.*

Problem 4. *Consider the wave equation*

$$u_{tt} = c^2 u_{xx},\ (x,t) \in \mathbb{R} \times (0,\infty)$$
$$u(x,0) = \phi(x)$$
$$u_t(x,0) = \psi(x).$$

(i) *(7 points) What is the domain of influence of a point $(x_0, 0)$?*

(ii) *(8 points) Suppose $\phi(x) = 0$ and $\psi(x) = 0$ if $|x| \geq R$, $R > 0$ a constant. Where (i.e. for what (x,t)) can you conclude that $u(x,t) = 0$? Where can you conclude that $u(x,t)$ is infinitely differentiable? Sketch these regions.*

(iii) *(12 points) Suppose that ϕ and ψ are as in (ii). Show that for any $R_0 > 0$, there exist $T > 0$ and a constant u_0 such that $|x| \leq R_0$, $t \geq T$ imply $u(x,t) = u_0$. (This means that for sufficiently large times t, u is constant over compact regions of space.) Also, express u_0 in terms of the initial conditions ϕ, ψ. You may use the explicit formula for the solution of this initial value problem:*

$$u(x,t) = \frac{1}{2}(\phi(x+ct) + \phi(x-ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds.$$