Math 140 Spring 2000 H. Wu

Final Exam: May 17, 2000.

1. Let $\alpha(s) = (r(s), 0, z(s))$ be a unit speed curve in the xz-plane such that $r > 0$, and let $\vec{x}(s,\theta) = (r(s)\cos\theta,\ r(s)\sin\theta,\ z(s))$ describe the simple surface (coordinate patch) M (where $-\pi < \theta < \pi$) obtained by revolving α around the z-axis. Show the following (you are allowed to make use of everything that precedes an item to do that item):

(a) Show that the metric coefficients $\{g_{ij}\}$ of M are given by the matrix $\begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$. (8 pts.)

(b) Define a geodesic on a surface. Use it to show that each s-curve on M (meridian) is a geodesic. (8 pts.)

(c) With $\vec{x}_s \equiv \dfrac{\partial \vec{x}}{\partial s}$, $\vec{x}_\theta \equiv \dfrac{\partial \vec{x}}{\partial \theta}$ as usual, compute the matrix of the second fundamental form:

$$\begin{pmatrix} \mathrm{II}(\vec{x}_s, \vec{x}_s) & \mathrm{II}(\vec{x}_s, \vec{x}_\theta) \\ \mathrm{II}(\vec{x}_\theta, \vec{x}_s) & \mathrm{II}(\vec{x}_\theta, \vec{x}_\theta) \end{pmatrix}$$ (14 pts.)

(d) Notation as in (c), compute the matrix of the Weingarten map L relative to $\{\vec{x}_s, \vec{x}_\theta\}$. (14 pts.)

(e) Show that the Gaussian curvature K and the mean curvature H satisfy $K = -\frac{\ddot{r}}{r}$ and
$H = \frac{1}{2r}(r\dot{r}\ddot{z} - r\dot{z}\ddot{r} + \dot{z})$. (8 pts.)

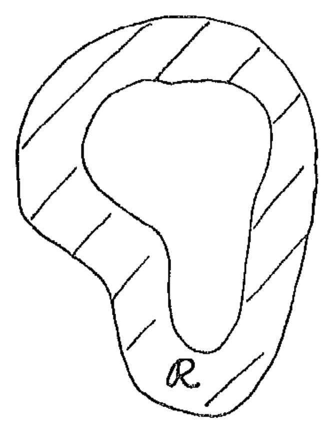
(f) Construct a surface of Gaussian curvature -1. (8 pts.)

2. (a) If every point of a simple surface M is an umbilic, show that M has constant principal curvature. (10 pts.)

(b) Use (a) to show that if a simple surface M has positive Gaussian curvature and each point of M is an umbilic, then M is part of a sphere. (10 pts.)

[Recall that every simple surface is <u>connected</u>.]

3. Let $\mathcal{R}$ be an "annulus" region (as shown) on a simple surface where both the inner and outer boundary curves are smooth. Starting with the Gauss-Bonnet theorem for triangles, compute



$$\int_{\mathcal{R}} K\,dA + \int_{\partial\mathcal{R}} \kappa_g\,ds$$

(K = Gaussian curvature, κ_g = geodesic curvature). (20 pts.)