MATH 54 FINAL

#1 If $\{f_1, f_2, f_3\}$ is a linearly independent set of functions from $\mathbb{R}$ to $\mathbb{R}$ then the Wronskian $W(f_1, f_2, f_3)(t) \neq 0$ for all t.

A. TRUE B. FALSE

#2 If $\{\vec{v_1}, \ldots, \vec{v_n}\}$ is a set of nonzero orthogonal vectors Then $\{\vec{v_1}, \ldots, \vec{v_n}\}$ is linearly independent.

A. TRUE B. FALSE

#3 Is The following set $\left\{ \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ orthonormal ?

A. TRUE B. FALSE

#4 Two independent real solutions to $x'' + 2x' + 2x = 0$ are:

A. e^t, e^{2t} B. $e^{(1+i)t}, e^{(1-i)t}$ C. $e^{-t}\sin t, e^{-t}\cos t$ D. $e^t\sin t, e^t\cos t$.

#5 The solution to the initial-value problem:

$$x'' + 2x' + 2x = 0, \quad x(0) = 0, \quad x'(0) = 1$$

A. Does not exist B. $C_1 e^{-t}\sin t + C_2 e^{-t}\cos t$ C. $e^{-t}\sin t$.

#6. Let $\mathcal{E}$ denote The standard basis for $\mathbb{R}^2$ i.e. $\mathcal{E} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ and $\mathcal{B} = \left\{ \begin{bmatrix} 5 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \end{bmatrix} \right\}$. The change of basis matrix $\underset{\mathcal{B} \leftarrow \mathcal{E}}{P}$ is:

A. $\begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}$ B. $\begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}$ C. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ D. None of These.

#7 The matrix $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ is diagonalizable ?

A. TRUE B. FALSE

#8 The system of ODEs $\begin{cases} y'' - 2x' + y = 0 \\ x'' - y' + x - y = 0 \end{cases}$ can be turned into the matrix equation $\vec{x}' = A\vec{x}$ where $\vec{x} = \begin{bmatrix} x \\ y \\ x' \\ y' \end{bmatrix}$, $A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & 1 \\ 0 & -1 & 2 & 0 \end{bmatrix}$

A. TRUE B. FALSE

#9 Given is the ODE $x'' + 2x' + 1 = te^{-t}$. The correct guess for x_p according to the method of undetermined coefficients is:

A. $(At+B)e^{-t}$ B. Ct^2e^{-t} C. $t^2(At+B)e^{-t}$ D. $t(At+B)e^{-t}$.

#10 Let $\vec{x_1}, \vec{x_2}, \vec{x_3}$ be 3 linearly independent vectors in $\mathbb{R}^n$ $(n \geq 3)$ and let $\vec{v_1}, \vec{v_2}, \vec{v_3}$ be orthogonal vectors obtained from the Gram-Schmidt process on $\{\vec{x_1}, \vec{x_2}, \vec{x_3}\}$ Then:

$$\vec{v_1} = \vec{x_1}, \quad \vec{v_2} = \vec{x_2} - \frac{\vec{x_2} \cdot \vec{v_1}}{\vec{v_1} \cdot \vec{v_1}} \vec{v_1}, \quad \vec{v_3} = \vec{x_3} - \frac{\vec{x_3} \cdot \vec{v_1}}{\vec{v_1} \cdot \vec{v_1}} \vec{v_1} - \frac{\vec{x_3} \cdot \vec{v_2}}{\vec{v_2} \cdot \vec{v_2}} \vec{v_2}$$

A. TRUE B. FALSE

#11 Suppose A: a 3×3 matrix has a basis of orthonormal eigenvectors $\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$. Let $Q = [\vec{v_1}\ \vec{v_2}\ \vec{v_3}]$. Which of the following matrix is always diagonal ?

A. $Q^T A Q$ B. QAQ^T C. QAQ^{-1} D. None of these

#12 The set $\left\{ \begin{bmatrix} a+2 \\ b-a \\ b-a+1 \end{bmatrix} \mid a,b \in \mathbb{R} \right\}$ is a subspace of $\mathbb{R}^3$?

A. TRUE B. FALSE

#13 Let $A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 1 & 3 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$

13.1 One least-square solution $\hat{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ to the linear equation $A\vec{x} = \vec{b}$ is obtained by solving $A\vec{x} = \vec{c}$ where $\vec{c}$ is:

A. $\begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$ B. $\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$ C. $\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$ D. $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

13.2 The normal equation to $A\vec{x} = \vec{b}$ is:

$$2x_1 + 5x_2 = 2$$
$$5x_1 + 13x_2 = 9$$

A. TRUE B. FALSE

#14 Let $A = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$. e^{At} is:

A. $\begin{bmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{bmatrix}$ B. $\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$ C. Neither A nor B

#15 Given is the ODE $x'' + x = \tan t$, $t \in (-\pi/2, \pi/2)$. According to the method of variation of parameters, a particular solution

$$x_p(t) = v_1(t)\cos t + v_2(t)\sin t$$

where $\vec{v_1}(t), \vec{v_2}(t)$ are:

A. $\int -\tan t \sin t \, dt, \int \sin t \, dt$ B. $\int -\frac{\sin^2 t}{\cos t} dt, \int \cos t \, dt$ C. $\int \sin t \, dt, \int -\frac{\sin^2 t}{\cos t} dt$.

\# 16 Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$ and $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^3$.

16.1 The matrix $\underset{\mathcal{B} \leftarrow \mathcal{B}}{[A]}$ is

A. $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$ B. $\begin{bmatrix} 6 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ C. $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

16.2 The eigenvalues for A are:

A. $\lambda = 1, 2, 3$ B. $\lambda = 0, 6$ C. $\lambda = -1, 2$

\# 17 Given is the ODE $x'' - 2x' + 2 = te^t \sin t$. According to the method of undetermined coefficients, a particular solution x_p has the form:

A. $Ate^t \sin t$ B. $t(At+B)e^t \sin t + t(Ct+D)e^t \cos t$ C. $t(At+B)e^t \sin t$

\# 18 Let $\vec{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$, then $\text{proj}_{\vec{u}} \vec{v} =$

A. $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ B. $\begin{bmatrix} 6/5 \\ 3/5 \\ 0 \end{bmatrix}$ C. $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ D. None of these.

\# 19 The general solution to $y''' - 2y'' - 5y' + 6y = 0$ is:

A. $c_1 e^x + c_2 e^{-2x} + c_3 e^{3x}$ B. $c_1 e^x + c_2 xe^x + c_3 x^2 e^x$ C. Neither

\# 20 Let A be an orthogonal $m \times m$ matrix. Then Row A = Col A

A. TRUE B. FALSE

#21 If $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$, $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. So $A\vec{u} = 6\vec{u}$. Then $e^{At}\vec{u}$ is:

A. $\begin{bmatrix} e^{6t} \\ 2e^{6t} \\ 3e^{6t} \end{bmatrix}$ B. $\begin{bmatrix} e^{At} \\ 2e^{At} \\ 3e^{At} \end{bmatrix}$ C. $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ D. None of These

#22 Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ and let $\vec{u} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ be a generalized eigenvector (but not an eigenvector) of A corresponding to $\lambda = 1$.

Then $e^{At}\vec{u} =$

A. $\begin{bmatrix} -2e^{t} \\ e^{t} \\ te^{t} \end{bmatrix}$ B. $\begin{bmatrix} -2e^{t} \\ e^{t} \\ 0 \end{bmatrix}$ C. $\begin{bmatrix} -2e^{At} \\ e^{At} \\ 0 \end{bmatrix}$ D. None of these.

#23 If $\vec{v_1}, \vec{v_2}$ are two linearly independent eigenvectors of A and $A\vec{v_1} = \lambda_1\vec{v_1}$, $A\vec{v_2} = \lambda_2\vec{v_2}$. Then: $\lambda_1 \neq \lambda_2$:

A. TRUE B. FALSE

#24 Let $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ be a one to one linear transformation.

Then $\dim(\text{range } T) =$

A. 2 B. 3 C. 1 D. 0

#25 Given That $y_1(t) = \frac{1}{4}\sin 2t$ is a solution to $y'' + 2y' + 4y = \cos 2t$ and that $y_2(t) = \frac{t}{4} - \frac{1}{8}$ is a solution to $y'' + 2y' + 4y = t$. Then a solution to $y'' + 2y' + 4y = 2t - 3\cos 2t$ is:

A. $-\frac{3}{4}\sin 2t + \frac{t}{2} - \frac{1}{4}$ B. $\frac{1}{4}\sin 2t + \frac{t}{4} - \frac{1}{8}$ C. $\sin 2t + t - 1$.

#26 (12 pts) Given is the initial-value problem:

$$y'' - 3y' + 2y = te^{2t} \qquad y(0) = 1,\ y''(0) = 1$$

Solve the problem using both methods: undetermined coefficients and Variation of parameters.

27 (12 pts) Solve the following initial value problem:

$$\vec{x}\,'(t) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \vec{x}(t) + \begin{bmatrix} e^{2t} \\ 0 \end{bmatrix} \qquad \vec{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

#28 (10 pts) Let $T: \mathbb{P}_2 \longrightarrow \mathbb{P}_2$ be

$$T(a_0 + a_1 t + a_2 t^2) = (a_0 - a_2) + (a_1 - 2a_2)t + a_0 t^2$$

and $\mathcal{B} = \{1, 1-t, t-t^2\}$ be a basis for dom (T), $\mathcal{C} = \{1, 1+t, 1+t^2\}$ be a basis for codomain (T). Compute $\underset{\mathcal{C} \leftarrow \mathcal{B}}{[T]}$ and $[T(1+t+t^2)]_{\mathcal{C}}$.

29 (12 pts) Let $\vec{v} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ and $H : x + y + z = 0$. Compute

$\text{proj}_H \vec{v}$ and $\text{dist}(\vec{v}, \text{proj}_H \vec{v})$.