UCB Math 128A-2, Summer 2009: Final Exam
Thursday 8/13

Name: ____________________________

- **Do not open until instructed to do so.**
- Books, notes, and calculators are not allowed.
- Justify all answers.
- Time limit is 120 minutes.
- If you get stuck on a (sub)problem, please *move on* and try it again later. Some of these problems are meant to be hard.

Grading

1.	/	10
2.	/	10
3.	/	10
4.	/	10
5.	/	10
6.	/	10
7.	/	10
8.	/	10
9.	/	10
10.	/	10
	/	100

Scratch/extra work

Scratch/extra work

1. (a) Prove that the bisection method on $[1, 2]$ will converge to a root of $x^3 - x - 1$.
   (b) Find a bound on the absolute error in approximating this root after n iterations.
   (c) How many iterations are necessary to ensure the relative error is at most 2^{-5}? (Your answer does not have to be exact.)

2. (a) Find an algebraic expression for the unique root (in $\mathbb{R}$) of $f(x) = x^2 - 2/x$.
   (b) Newton's method searches for this root using the iteration $p_{n+1} = g(p_n)$. Find $g(x)$. Simplify your answer.
   (c) Is the convergence linear? Is it at least quadratic?
   (d) Find the exact order of convergence.

3. (a) Estimate $y(1/2)$ with polynomial interpolation given the data $y(0) = 1$, $y(1) = 2$.
   (b) Repeat part (a) with Hermite interpolation, if $y(t)$ is a solution to $y' = y$.

4. (a) Transform $\int_{-3}^{3} \frac{1}{t^2+1}\, dt$ to an integral of the form $\int_{-1}^{1} f(x)\, dx$.
   (b) Find the coefficients below for the three-point Gaussian quadrature rule:

$$\int_{-1}^{1} f(x)\, dx \approx af\left(-\sqrt{\frac{3}{5}}\right) + bf(0) + cf\left(+\sqrt{\frac{3}{5}}\right).$$

   (c) What is this rule's degree of precision?
   (d) Estimate the integral from (a) using this rule. Express your answer as a fraction.

5. (a) Find the local truncation error for the method $w_{i+1} = 4w_i - 3w_{i-1} - 2hf(t_{i-1}, w_{i-1})$.
   (b) Classify this multistep method as strongly stable, weakly stable, or unstable.

6. Describe (algebraically) the region of absolute stability for *Heun's Method*,

$$w_{i+1} = w_i + \frac{h}{4}[f(t_i, w_i) + 3f(t_i + \frac{2}{3}h, w_i + \frac{2}{3}hf(t_i, w_i))].$$

7. (a) Explain how the second-order IVP $y''+4y'-5y=0$, $y(0)=2$, $y'(0)=-3$ is equivalent to this system of first-order IVPs:

$$\mathbf{u}' = \begin{bmatrix} 0 & 1 \\ 5 & -4 \end{bmatrix} \mathbf{u}, \quad \mathbf{u}(0) = \begin{bmatrix} 2 \\ -3 \end{bmatrix}.$$

(b) If A is the above matrix, find $\|A\|_\infty$.

8. Solve this system via elimination *with partial pivoting* followed by back substitution.

$$\left[\begin{array}{rrr|r} 1 & -1 & 0 & 0 \\ -2 & 5 & 1 & 2 \\ -1 & 4 & 5 & -2 \end{array}\right]$$

9. Let $A = \begin{bmatrix} 4 & 2 & 0 & 0 \\ 2 & 5 & 2 & 0 \\ 0 & 2 & 5 & 2 \\ 0 & 0 & 2 & 5 \end{bmatrix}$.

(a) Is A strictly diagonally dominant?

(b) Perform elimination with *no* pivoting. Compute $\det(A)$ using the result.

(c) Is A a symmetric, positive definite matrix?

10.

Find a Cholesky (LL^*) factorization for this matrix: $A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 2 \\ 1 & 0 & 2 & 1 \\ 1 & 2 & 1 & 6 \end{bmatrix}.$