# Math 104: Introduction to Analysis

## Midterm March 20th, 2002

Weingart

Name: ______________________________

Signature: ___________________________

There are 9 problems on this midterm worth 100 points of 400 for the class in total. The first 5 problems are each worth 8 points for the correct answer, whereas the last 4 problems are more difficult and worth 15 points each. You must show your work to get any credit for the last 4 problems. Successful midterm!

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | Total |
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**Problem 1:** (8 points)
Recall that 0 is called a limiting point of a sequence $(s_n)_{n\geq 1}$ if for every $\varepsilon > 0$ there are infinitely many $n \in N$ with $|s_n| < \varepsilon$. If however 0 is NOT a limiting point for a sequence $(s_n)_{n\geq 1}$, what do you conclude?

- ☐ There is some $\varepsilon > 0$ such that $|s_n| \geq \varepsilon$ for all but finitely many $n \in \mathbb{N}$.
- ☐ There is some $\varepsilon > 0$ such that $|s_n| \geq \varepsilon$ for infinitely many $n \in \mathbb{N}$.
- ☐ For all $\varepsilon > 0$ there are infinitely many $n \in \mathbb{N}$ with $|s_n| \geq \varepsilon$.

**Problem 2:** (8 points)
Every rational number can be written in lowest possible terms $\frac{p}{q}$, so that $p$ and $q$ have no common divisor and $q > 0$. Consider the function $f : \mathbb{Q}\cap[0,1] \longrightarrow \mathbb{R}$ defined on the rational numbers $r \in [0,1]$ by writing $r = \frac{p}{q}$ in lowest possible terms and setting $f(r) := \frac{1}{q}$.

- ☐ There are different continuous functions $g : [0,1] \longrightarrow \mathbb{R}$ extending $f$ such that $g(r) = f(r)$ for all rational numbers $r$ in $[0,1]$.
- ☐ There is exactly one continuous function $g : [0,1] \longrightarrow \mathbb{R}$ extending $f$ such that $g(r) = f(r)$ for all rational numbers $r$ in $[0,1]$.
- ☐ There is no continuous function $g : [0,1] \longrightarrow \mathbb{R}$ extending $f$ such that $g(r) = f(r)$ for all rational numbers $r$ in $[0,1]$.

**Problem 3:** (8 points)
Consider an interval $I \subset \mathbb{R}$ and some continuous function $f : I \longrightarrow \mathbb{R}$ defined on $I$. Which of the following statements is true?

- ☐ If $I = [a,b]$ is a closed interval then $f(I) \subset \mathbb{R}$ is a closed interval for every continuous function $f$ defined on $I$.
- ☐ If $I = [a,\infty)$ is a closed interval then $f(I) \subset \mathbb{R}$ is a closed interval for every continuous function $f$ defined on $I$.
- ☐ If $I = (a,b)$ is an open interval then $f(I) \subset \mathbb{R}$ is an open interval for every continuous function $f$ defined on $I$.

**Problem 4:** (8 points)
This impressive list of names and well–sounding statements connected with them contains one flawed reformulation, namely?

- ☐ By the Theorem of Heine–Borel every closed subset $A$ of $\mathbb{R}$ is compact or unbounded.
- ☐ By Banach's Fixed Point Theorem every contraction $f: S \longrightarrow S$ of a Cauchy–complete metric space $S$ has a unique fixed point.
- ☐ By the Theorem of Bolzano–Weierstraß every bounded sequence $(s_n)_{n\geq 1}$ of real numbers has a convergent subsequence.

**Problem 5:** (8 points)
Calculating the radius of convergence of a power series can be tricky. Unluckily I was careless and made an error in my calculations, please find it:

- ☐ The radius of convergence of the power series $\sum_{m=0}^{\infty} (m+1)\, x^m$ is $R = 1$.
- ☐ The radius of convergence of the power series $\sum_{m=0}^{\infty} (\frac{1}{4})^m\, x^{2m}$ is $R = 2$.
- ☐ The radius of convergence of the power series $\sum_{m=0}^{\infty} \binom{2m}{m}\, x^m$ is $R = 4$.

**Problem 6:** (15 points)
Formulate the completeness axiom of the real numbers and write down the definition of the limit of a convergent sequence of real numbers.

**Problem 7:** (15 points)
Show that the two functions $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ and $g : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by $f(x,y) := x$ and $g(x,y) := y$ are continuous. Conclude that the function $f^2g$ defined by $(f^2g)(x,y) := x^2y$ is continuous as well.

**Problem 8:** (15 points)
Consider a closed subset $A \subset S$ of a metric space $S$, in other words its complement $S \setminus A$ is open. Prove explicitly that every convergent sequence $(a_n)_{n\geq 1}$ of elements $a_n \in A$ converges to a limit $\lim_{n\to\infty} a_n$ in $A$.

**Problem 9:** ( 15 points )
Consider a closed interval $[0,1] \subset \mathbb{R}$ and the set of all continuous functions on $[0,1]$:

$$C^0(\,[0,1]\,) \;:=\; \{\, f : [0,1] \longrightarrow \mathbb{R} \mid f \text{ is continuous } \}\,.$$

Verify the axioms of a metric space for the following distance function on $C^0([0,1])$:

$$\operatorname{dist}(\, f,\, g\,) \;:=\; \sup\{\; |f(x) - g(x)| \;\mid x \in [0,1]\,\} \qquad f, g \in C^0([0,1])\,.$$

Note that for fixed $f, g \in C^0([0,1]$ the supremum in the definition of $\operatorname{dist}(f,g)$ is always a real number and never $+\infty$. Why?