On the optimal density of sphere packings in $\mathbb{E}^8$ and the uniqueness theorem on finite packings with optimal density

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Abstract:Let $\mathcal{P}$ be a finite cluster of unit spheres and $\mathcal{P}^*$ be an extension of $\mathcal{P}$ which makes all the local cells (i.e. Voronoi cells) of $S_j \in \mathcal{P}$, $\{C(S_j, \mathcal{P}^*)\}$, bounded. Then, it is natural to define the *relative density* $\rho(\mathcal{P}, \mathcal{P}^*)$ by setting

(1) $$\rho(\mathcal{P}, \mathcal{P}^*) = \sum_{S_j \in \mathcal{P}} \mathrm{vol}\,(S_j) \Big/ \sum_{S_j \in \mathcal{P}} \mathrm{vol}\, C(S_j, \mathcal{P}^*)$$

and to define the *intrinsic density* of $\mathcal{P}$ to be the least upper bound of $\{\rho(\mathcal{P}, \mathcal{P}^*)\}$ for all possible extensions of $\mathcal{P}$, namely

(2) $$\rho(\mathcal{P}) = l.u.b.\,\{\rho(\mathcal{P}, \mathcal{P}^*), \mathcal{P} \subset \mathcal{P}^*\}$$

In the very special case that $\mathcal{P} = \{S\}$ consists of a single sphere, the above *intrinsic density* $\rho(\{S\})$ is exactly the *optimal local density* of sphere packings in $\mathbb{E}^8$. Anyhow, the following are two fundamental problems in the study of sphere packings in $\mathbb{E}^8$, namely

Problem 1:What is the *optimal local density* in $\mathbb{E}^8$? and what are the geometric structures of those *tightest* local packings (i.e. the ones with optimal local density) ?

Problem 2:What is the *optimal intrinsic density* for cluster of unit spheres in $\mathbb{E}^8$ of a given cardinality, namely

(3) $$\rho_N = l.u.b.\{\rho(\mathcal{P}), \#(\mathcal{P}) = N\} = ?$$

Moreover, what are the geometric structures of those N-clusters with $\rho(\mathcal{P}) = \rho_N$?

It is a remarkable, pleasant surprise that both of the above two problems have clean-cut solutions and the strongest possible uniqueness theorems, namely

Theorem I:The optimal local density of sphere packings in $\mathbb{E}^8$ is equal to $\pi^4/384$. The local density of a local packing $\mathcal{L}(S_0)$ is equal to $\pi^4/384$ when and only when $\mathcal{L}(S_0)$ is isometric to the local packing type of the lattice packing associated to the root lattice of E_8 (i.e. the exceptional Lie group of rank 8).

Theorem II:Let $\mathcal{P}$ be a finite cluster of identical spheres in $\mathbb{E}^8$. Then

(4) $$\rho(\mathcal{P}) \leq \frac{\pi^4}{384}$$

and equality holds when and only when $\mathcal{P}$ is isometric to a finite cluster in the E_8-lattice packing (up to a scaling). [A finite packing is called a *cluster* if any pair of them can be linked by a chain with consecutive center-distances less than 2√2-time of the radius.]

The purpose of this talk is to present the proofs of the above two theorems.
[In fact, Theorem II follows readily from Theorem I.]