Final Exam for **MATH 104**, Section 1

Fall 2008, December 15, UC Berkeley

Problem 1 [20P]

DEFINITIONS and THEOREMS.

(a) Give three different but equivalent definitions of a *continuous function* $f : M \to N$ between metric spaces M and N.

(b) If $f_n, f : [a, b] \to \mathbb{R}$, define what it means that (f_n) *converges uniformly to* f, $f_n \rightrightarrows f$.

(c) State the *Riemann-Darboux Integrability Criterion.*

(d) State the *Fundamental Theorem of Calculus.*

Problem 2 [20P]

EXAMPLES. Give an example of

(a) a compact subset of $\mathbb{R}^2$ that is neither homeomorphic to the closed unit disk $\mathbb{B} = \{x \in \mathbb{R}^2 : \|x\| \leq 1\}$ nor to the closed interval $[0, 1]$.

(b) a function $f : \mathbb{R} \to \mathbb{R}$ that is twice differentiable but such that f'' is not continuous.

(c) a power series $\sum c_k x^k$ with radius of convergence 2 such that the series converges for $x = -2$ but does not converge for $x = 2$.

(d) a sequence (f_n) of continuous functions $f_n : [0, 1] \to \mathbb{R}$ that does not have a convergent subsequence in $\mathcal{C}^0([0, 1], \mathbb{R})$ with respect to the sup-metric. (Hence $\mathcal{C}^0([0, 1], \mathbb{R})$ is not compact.)

You do not have to justify your examples, just state them.

Problem 3 [15P]

Let (M, d) be a metric space.

(a) Show that the union of finitely many compact sets $K_1, \ldots, K_n \subseteq M$ is compact.

(b) Suppose $K \subseteq M$ is compact and $f : K \to \mathbb{R}$ is continuous. Show that $Z_f = \{x \in K : f(x) = 0\}$ is compact.

Problem 4 [15P]

Let (M, d) be a metric space. Given a set $S \subseteq M$, define the *characteristic function* $\chi_S : M \to \{0, 1\}$ of S as

$$\chi_S(x) = \begin{cases} 1 & \text{if } x \in S, \\ 0 & \text{if } x \notin S. \end{cases}$$

(a) Recall that the *boundary of* S, ∂S, is defined as $\partial S = \lim(S) \cap \lim(M \setminus S)$. Show that χ_S is discontinuous at x if and only if $x \in \partial S$.

(b) Infer that the characteristic function of the Cantor set (as a subset of $[0, 1]$) is Riemann-integrable.

Problem 5 [15P]

Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous.

(a) Show that if f is differentiable and the derivative f' is bounded, then f is uniformly continuous.

(b) Given $a < b$, argue that f is Riemann integrable on $[a, b]$ and show that there exists an $x \in [a, b]$ such that

$$f(x) = \frac{1}{b - a} \int_a^b f(t)dt.$$

Problem 6 [15P]

Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined as

$$f(x) = \sum_{k=0}^{\infty} \frac{\sin(2kx)}{2^k}.$$

(a) Argue that this function is well-defined, i.e. that for each $x \in \mathbb{R}$, $f(x)$ exists and is finite.

(b) Show that f is Riemann integrable on any interval $[a, b]$.

(c) Compute

$$\int_0^{\pi} f(x)dx.$$

Extra Credit.
In a metric space M, the *interior* of a set $S \subseteq M$, $\text{int}(S)$, is defined as the set of all points $s \in S$ for which $M_r(s) \subseteq S$ for some $r > 0$. If S is connected, is $\text{int}(S)$ connected, too? Prove or give a counterexample.