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Fall 2008, Math H110
**Final Exam**

17 Dec., 2008
5:00-8:00 PM

This is a closed book closed notes exam.

Attempt all problems. Write solutions on these sheets. Ask for scratch paper if the fronts and backs of these pages are not sufficient; put your name on any such extra sheets and hand them in with your exam.

Credit for an answer may be reduced if a large amount of irrelevant or incoherent material is included along with the correct answer.

Questions begin on the next sheet. Fill in your name on this sheet now, but do not turn the page until the signal is given. At the end of the exam, stop writing and close your exam as soon as the ending signal is given, or you will lose points.

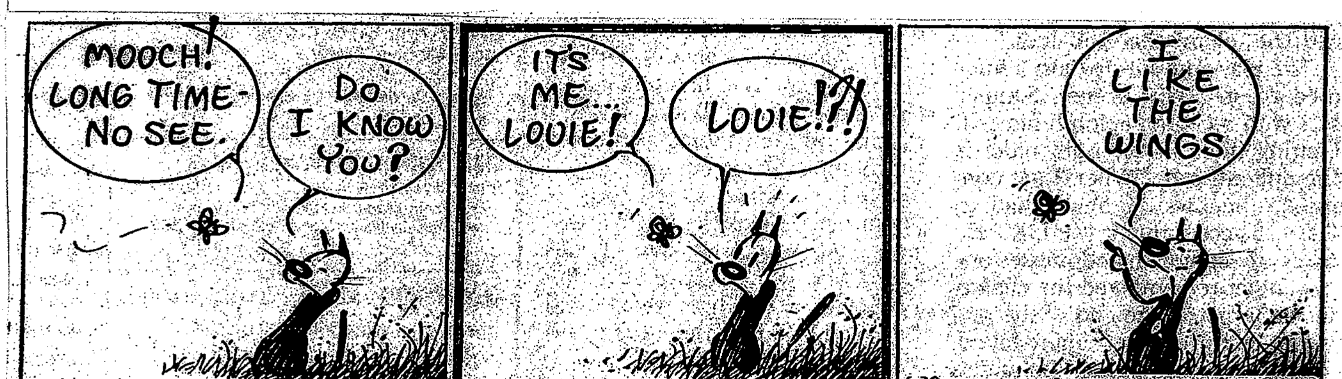
Think clearly, stay calm.

**Your name** ________________________________

*Leave blank for grading*

| | |
|---|---|
| 1 | / 8 |
| 2 | / 8 |
| 3(a-c) | / 15 |
| 3(d-f) | / 15 |
| 4 | / 20 |
| 5(a) | / 10 |
| 5(b) | / 10 |
| 5(c) | / 10 |
| 6 | / 4 ( + *possible extra credit*) |
| Σ | /100 |

**MUTTS** *Patrick McDonnell*



**1.** (8 points: 4 points each.) Find the following.

*work* | *answers:*

(a) A linear transformation $T: \mathbb{R}^3 \to R$ such that $N(T) = \text{span}(\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \})$. You may express your answer either by giving the matrix $A$ for $T$ with respect to the standard bases of $\mathbb{R}^3$ and $R$, or by giving a formula for $T(\begin{pmatrix} x \\ y \\ z \end{pmatrix})$.

(a)

(b) The Jordan canonical form $J$ of the matrix $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ over $C$, and a matrix $Q$ such that $J = Q^{-1} A\, Q$.

(b)

**2.** (8 points.) Find the singular value decomposition of the matrix $A = \begin{pmatrix} 2 & 3 \\ 0 & 2 \\ 0 & 0 \end{pmatrix}$ over $R$. Precisely, find bases $\beta = \{v_1, v_2\}$ of $\mathrm{R}^2$ and $\gamma = \{w_1, w_2, w_3\}$ of $\mathrm{R}^3$ which are orthonormal with respect to the standard inner products on those spaces, and such that multiplication by $A$ takes each member of $\beta$ to a positive scalar multiple of a member of $\gamma$; and give the scalars $\sigma_i$ in question (the singular values of $A$). Give your answers where indicated at the bottom of the page.

$v_1$ $\begin{pmatrix} \\ \\ \end{pmatrix}$ $v_2$ $\begin{pmatrix} \\ \\ \end{pmatrix}$ $w_1$ $\begin{pmatrix} \\ \\ \\ \end{pmatrix}$ $w_2$ $\begin{pmatrix} \\ \\ \\ \end{pmatrix}$ $w_3$ $\begin{pmatrix} \\ \\ \\ \end{pmatrix}$

*Singular values:*

**3.** (30 points, 5 points each.) Complete the following definitions. You may use, without defining them, any terms or symbols that our text defines before defining the word or symbol asked for. Your definitions do not have to have exactly the same wording as those in the text, but for full credit they should be clear, and mean the same thing as those definitions. More space is provided for the answers than a concise answer is likely to need.

(a) If $A$ is a ______________________________ matrix over a field $F$, then the **characteristic polynomial** $f(t)$ of $A$ means ______________________________

______________________________________________________________.

(b) If $A$ is a ______________________________ matrix over a field $F$, then the **minimal polynomial** $f(t)$ of $A$ means ______________________________

______________________________________________________________.

(c) If $\mathsf{T}$ is a linear operator on a vector space $\mathsf{V}$, and $\lambda$ is an eigenvalue of $\mathsf{T}$, then a **generalized eigenvector** of $\mathsf{T}$ means an nonzero element $v \in \mathsf{V}$ such that

______________________________________________________________

______________________________________________________________.

(3, continued)

(d) If $A$ is a regular $n \times n$ transition matrix, then the **fixed probability vector** or **stationary vector** of $A$ means the unique ______________________________

______________________________________________.

(e) If $\mathsf{V}$ and $\mathsf{W}$ are inner product spaces, and $\mathsf{T} \colon \mathsf{V} \to \mathsf{W}$ is a linear transformation, then the adjoint of $\mathsf{T}$ (if it exists) means the unique linear transformation $\mathsf{T}^*$: __________ $\to$ __________ such that ________________

______________________________________________.

(f) A linear operator on a real inner product space is said to be **orthogonal** if

______________________________________________

______________________________________________.

**4.** (20 points, 5 points each.) For each of the items listed below, either *give an example* with the property stated, or give a brief reason why *no such example exists.*

If you give an example, you do *not* have to prove that it has the property stated; however, your examples should be specific; i.e., even if there are many objects of a given sort, you should name a particular one. If you give a reason why no example exists, don't worry about giving reasons for your reasons; a simple statement will suffice.

(a) A square matrix $A$ whose characteristic polynomial splits, but for which there does not exist a basis of eigenvectors.

(b) An inner product space V, and a projection map T on V that is not self-adjoint.

(c) A self-adjoint operator on a complex inner product space having the complex number $i$ as an eigenvalue.

(d) A bilinear form $H$ on a real vector space V such that $H$ is not an inner product.

**5.** (30 points, 10 points each) Prove the following statements. You may assume all results proved in our text.

(a) Let $A$ be an $n \times n$ matrix over a field $F$. Show that

$$\dim(\mathrm{span}(\{I_n, A, A^2, \ldots\})) \leq n.$$

(Suggestion: Show that the span in question equals $\mathrm{span}(\{I_n, A, A^2, \ldots A^{n-1}\})$.)

(5, continued)

(b) Let T be a linear operator on a finite-dimensional real or complex inner product space V. Show that if V has an orthonormal basis of eigenvectors of T, then T is a normal operator; i.e., satisfies $T^*T = TT^*$. (For complex inner product spaces, this is a converse to a result in our text.)

(5, continued)

(c) Let $H$ be a bilinear form on a vector space V of finite dimension $n > 1$ over a field $F$. Show that for every $x \in V$ there exists a nonzero $y \in V$ with $H(x, y) = 0$.

**6.** (4 points, plus up to 10 points extra credit.) Let $T$ be a linear operator on a nonzero vector space $V$ over a field $F$.

(a) (4 points). Show that if $T$ has the form $\lambda I$ for some $\lambda \in F$, then every subspace $W \subseteq V$ is T-invariant (i.e., satisfies $T(W) \subseteq W$).

(b) (extra credit up to 10 points). Conversely, show that if every subspace $W \subseteq V$ is T-invariant, then $T = \lambda I$ for some $\lambda \in F$.