# Math 74, Sec. 2, Instructor: Walter Kim
## Final Exam (80 pts.), Tuesday, December 20, 2005

Name:

| Problem | Max. points | Your Points |
|---|---|---|
| 1 | 10 | |
| 2 | 10 | |
| 3 | 10 | |
| 4 | 10 | |
| 5 | 10 | |
| 6 | 10 | |
| 7 | 10 | |
| 8 | 10 | |
| Total | 80 | |

1. (10 pts.) (a) State the Well-Ordering Principle.

(b) Prove using the Well-Ordering Principle that $\sqrt{2}$ is not in $\mathbb{Q}$. Here by $\sqrt{2}$ we mean the positive real number satisfying the equation $x^2 - 2 = 0$. (Some hints: Keep in mind that since $1 < 2 < 4$, we know that $1 < \sqrt{2} < 2$. You will probably want to consider the set of positive integers $n$ such that $n\sqrt{2}$ is an integer.)

2. (10 pts.) (a) State the Chinese Remainder Theorem.

(b) Is there a solution for the equations $x \equiv 1 \pmod 2$ and $x \equiv 2 \pmod 4$? Why does this not contradict the Chinese Remainder Theorem?

(c) Is there a solution for the equations $x \equiv 1 \pmod 2$ and $x \equiv 3 \pmod 4$?

2. (con't) (d) Prove the existence of the solution in the Chinese Remainder Theorem. (You are not asked to prove the uniqueness part.)

(scratch paper)

3. (10 pts.) Consider the map $\varphi : \mathbb{Z}/5\mathbb{Z} \to \mathbb{Z}/5\mathbb{Z}$ given by $\varphi([n]) = [2n]$.

(a) Write down where $\varphi$ takes each of the elements of $\mathbb{Z}/5\mathbb{Z}$.

(b) Show that $\varphi$ preserves addition.

(c) Show that $\varphi$ DOES NOT preserve multiplication.

4. (10 pts.) Let $m$ be an integer $\geq 2$.

(a) Show that addition in $\mathbb{Z}/m\mathbb{Z}$ is commutative. (You can use any properties you know about $\mathbb{Z}$.)

(b) Show that if $m$ is prime, then all elements of $\mathbb{Z}/m\mathbb{Z}$ that are not the 0 element have a multiplicative inverse.

4. (con't) (c) Show that if $m$ is not prime, then $\mathbb{Z}/m\mathbb{Z}$ is not a field. (Hint: Find a nonzero element of $\mathbb{Z}/m\mathbb{Z}$ that does not have a multiplicative inverse. Maybe a divisor of $m$ will work.)

5. (10 pts.) Let $F$ be a field.
(a) Let $x \in F$ with $x \neq 0$. Show that the multiplicative inverse of $x$ is unique.

(b) Show that $0 \cdot x = 0$ for all $x \in F$. (Hint: Use the fact that $0 + 0 = 0$ which is true because why?)

6. (10 pts.) Consider the sequence

$$2, 4, 10, 28, 82, 244, \ldots$$

given by $x_n = 3^n + 1$.

(a) Does the sequence $(x_n)_{n \in \mathbb{N}}$ converge with respect to the 3-adic absolute value? If so, to what? Prove your answer.

6. (con't) We will now show together that the sequence $(x_n)_{n\in\mathbb{N}}$ given by $x_n = 3^n + 1$ does not converge with respect to the usual absolute value. We will do this by contradiction. Suppose by contradiction that $(x_n)_{n\in\mathbb{N}}$ converges to $L \in \mathbb{Q}$. We can write $L = \frac{a}{b}$ where $a \in \mathbb{Z}$, $b \in \mathbb{N}^*$ and $gcd(a, b) = 1$.

(b) Show that there is some $c \in \mathbb{N}$ such that for all $n \geq c$, $|x_n - L| > 1/100$. (Hint: $c = |a|$ works. Show why this is true and you are done with this part. Look at the cases when $a \geq 0$ and $a < 0$.)

(c) What is the definition of $(x_n)_{n\in\mathbb{N}}$ conveges to $L \in \mathbb{Q}$. I'll give you part of it: Given $p \in \mathbb{N}^*$, there exists an $m \in \mathbb{N}$ such that (fill in the big blank) $< 1/p$.

6. (con't) So we know from part (b) that there is some $p$, namely $p = 100$ such that there is no $m$ that works. This contradicts the assumption that $(x_n)_{n\in\mathbb{N}}$ conveges to $L$. Thus $(x_n)_{n\in\mathbb{N}}$ does not converge to anything.

(d) Explain why there is no $m$ that works for $p = 100$. (Hint: If you have an $m \in \mathbb{N}$, there is some $n \geq m$ such that $n \geq c$.)

7. (10 pts.) Let $p$ be a prime. Show that the $p$-adic absolute value $|\cdot|_p$ on $\mathbb{Q}$ satisfies

$$|x+y|_p \leq \max(|x|_p, |y|_p)$$

for all $x, y \in \mathbb{Q}$. I will start the proof for you: Let $x, y \in \mathbb{Q}$. We can write $x = \frac{c}{d} \cdot p^\alpha$ where $c \in \mathbb{Z}$, $d \in \mathbb{N}^*$, $\alpha \in \mathbb{Z}$, $p$ does not divide $c$, and $p$ does not divide $d$. We can write $y = \frac{e}{f} \cdot p^\beta$ where $e \in \mathbb{Z}$, $f \in \mathbb{N}^*$, $\beta \in \mathbb{Z}$, $p$ does not divide $e$, and $p$ does not divide $f$. (Hint: Evaluate $|x|_p$, $|y|_p$, and $|x+y|_p$. Then look at the three cases: $\alpha > \beta$, $\alpha > \beta$, $\alpha = \beta$.)

8. (10 pts.) (a) Show that every $z \in \mathbb{C}$ with $z \neq 0$ has a multiplicative inverse. In otherwords, find the multiplicative inverse of $z = x + iy \in \mathbb{C}$.

(b) Show that every $z = x + iy \in \mathbb{C}$ has a square root. In otherwords, find an element $w = u + iv \in \mathbb{C}$ such that $w^2 = z$.

(c) Show that every polynomial of the form $ax^2 + bx + c$ where $a, b, c \in \mathbb{C}$ has a root in $\mathbb{C}$. (Hint: $(-b + \sqrt{b^2 - 4ac})(2a)^{-1}$ is a root. Argue why this is a complex number if $a$, $b$, and $c$ are complex numbers.)