Math 113   Introduction to Abstract Algebra—Prof. Haiman   Fall, 2003

Final Exam

Name _____

Instructions:

1. Please do not look at the problems until everyone has an exam and you have been told to begin.

2. Write answers on the exam itself, attaching extra sheets if necessary. Turn in only work you wish to have graded; do not include scratch work.

3. Books, notes, calculators or other aids may not be used.

4. Show enough work to make your reasoning clear, even if a problem has a true/false or numerical answer.

5. There are 9 problems, with value indicated on each problem.

6. Time: 3 hours.

| For grading only			
1		2	
3		4	
5		6	
7		8	
9			
Total:			

1. (12 pts.) Which of the groups listed below are isomorphic to which others?

(a) $\mathbb{Z}_{20}$ (b) $\mathbb{Z}_{10} \times \mathbb{Z}_2$ (c) $\mathbb{Z}_5 \times \mathbb{Z}_4$
(d) $\mathbb{Z}_5 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ (e) D_{10}

2. (12 pts.) (a) Let H_1 and H_2 be distinct subgroups of a group G, with $|H_1| = |H_2| = p$, where p is prime. Show that $H_1 \cap H_2 = \{1\}$.

(b) Determine how many subgroups of order 5 the permutation group S_5 has.

3. (10 pts.) The *icosahedron* is the regular solid in 3 dimensions with 20 triangular faces, sketched below.
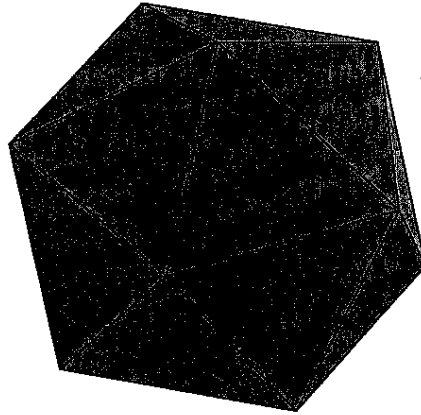
Let G be the group of rotational symmetries of the icosahedron.

(a) What is the order of the isotropy subgroup of a face of the icosahedron?

(b) What is the order of G?

4. (12 pts.) Recall that a group is *simple* if it has no proper non-trivial normal subgroup. For each of the following possibilities, either give an example of a simple group of that order or show that none exists.

    (a) order 4;

    (b) order 17;

    (c) order 60.

5. (10 pts.) Decide whether or not

$$\frac{x+1}{x+2} = \frac{x^2+2}{x^2+x+1}$$

is true in a field of quotients of the integral domain $\mathbb{Z}_3[x]$.

6. (12 pts.) Decide whether or not each of the following subsets $I \subseteq \mathbb{Z}[x]$ is an ideal. If so, describe a more familiar ring to which the factor ring $\mathbb{Z}[x]/I$ is isomorphic.

(a) $I = \{f(x) : \text{the constant term of } f(x) \text{ is a multiple of } 3\}$;

(b) $I = \{f(x) : \text{the coefficient of } x^2 \text{ in } f(x) \text{ is zero}\}$.

7. (10 pts.) Let R be a commutative ring with unity. Prove that the ideal $(x) \subseteq R[x]$ is prime if and only if R is an integral domain.

8. (10 pts.) Let α be a zero of $x^3 - 2$ in a field extension $\mathbb{Q}(\alpha)$ of $\mathbb{Q}$. Find the rational numbers a, b, c such that $(1 + \alpha)^{-1} = a + b\alpha + c\alpha^2$.

9. (12 pts.) Let $\sqrt[3]{2}$ be the real cube root of 2 and let i be the imaginary unit in $\mathbb{C}$, as usual.
   (a) Find $\mathrm{irr}(\sqrt[3]{2}, \mathbb{Q})$, $\mathrm{irr}(i, \mathbb{Q})$ and $\mathrm{irr}(i, \mathbb{Q}(\sqrt[3]{2}))$.
   (b) Compute the degree of the extension $[\mathbb{Q}(\sqrt[3]{2}, i) : \mathbb{Q}]$.
   (c) Show that $\mathbb{Q}(i\sqrt[3]{2}) = \mathbb{Q}(i, \sqrt[3]{2})$.
   (d) Find $\mathrm{irr}(i\sqrt[3]{2}, \mathbb{Q})$.