MATHEMATICS 170 — FINAL EXAM FALL 03 EVANS

Problem 1. (i) State the Brouwer Fixed Point Theorem.

(ii) State the Kakutani Fixed Point Theorem.

Problem 2. (i) State the dual problem (D) associated with the primal linear programming problem:

$$(P) \qquad \text{minimize } c \cdot x, \quad \text{subject to } Ax = b,\ x \geq 0.$$

(ii) Prove that if x is feasible for (P), y is feasible for (D) and $c \cdot x = b \cdot y$, then x is optimal for (P) and y is optimal for (D).

Problem 3. Find the value for the two-person, zero-sum game associated with the payoff matrix

$$A = \begin{bmatrix} 4 & 0 \\ 6 & 3 \end{bmatrix}.$$

(Hints: The associated linear programming problems are

$$(P) \qquad \text{minimize } \sum u_i, \text{ subject to } A^T u \geq 1, u \geq 0,$$

$$(D) \qquad \text{maximize } \sum v_j, \text{ subject to } Av \leq 1, v \geq 0.)$$

Problem 4. (i) Let f be a flow on a capacitated network (N, k), and let (C, C') denote a cut. Define the expressions $v(f)$, $f(C, C')$, $k(C, C')$ and then show that

$$f(C, C') \leq k(C, C').$$

(ii) Suppose that

$$v(f_0) = k(C_0, C_0')$$

for some flow f_0 and some cut (C_0, C_0'). Show that f_0 is a maximal flow and (C_0, C_0') is a minimal cut.

(Hint: You may use the fact that for any cut and for any flow, we have $v(f) = f(C, C')$.)

Problem 5. Set up and use the simplex tableau to do a Phase II calculation for the problem of minimizing $x_1 - x_2 - x_3$, subject to

$$x \geq 0, \quad \begin{bmatrix} 2 & -3 & 1 \\ 5 & -9 & 4 \end{bmatrix} x = \begin{bmatrix} 2 \\ 7 \end{bmatrix}.$$

Start with $x^0 = \frac{1}{3}[1\ 0\ 4]^T$. What does Phase II tell you?

Problem 6. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth convex function. Show that

$$(\nabla f(x) - \nabla f(y)) \cdot (x - y) \geq 0 \qquad \text{for all } x, y.$$

(Hint: First write down the inequality (presented in class) that says the graph of f lies above the tangent plane to the graph at any point.)

Problem 7. Suppose X, Y are closed, bounded sets and let $f : X \times Y \to \mathbb{R}$ be a continuous function.

(i) Show that

$$\max_{x \in X} \min_{y \in Y} f(x, y) \leq \min_{y \in Y} \max_{x \in X} f(x, y).$$

(ii) Suppose there exists a saddle point $(x_0, y_0) \in X \times Y$, satisfying

$$f(x, y_0) \leq f(x_0, y_0) \leq f(x_0, y)$$

for all $x \in X, y \in Y$. Show that then

$$\max_{x \in X} \min_{y \in Y} f(x, y) = \min_{y \in Y} \max_{x \in X} f(x, y)$$

Problem 8. Recall that $L(x, u) = g(x) + u \cdot \mathbf{f}(x)$ is the Lagrangian function associated with the nonlinear programming problem

(MAX) maximize $g(x)$, subject to the constraints $\mathbf{f}(x) \geq 0, x \geq 0$.

The Kuhn-Tucker differentiability conditions are

$$(D) \qquad \begin{cases} \nabla_x L(x^0, u^0) \leq 0, \ x^0 \cdot \nabla_x L(x^0, u^0) = 0, \ x^0 \geq 0 \\ \nabla_u L(x^0, u^0) \geq 0, \ u^0 \cdot \nabla_u L(x^0, u^0) = 0, \ u^0 \geq 0. \end{cases}$$

For the particular problem of maximizing $-8x_1^2 - 10x_2^2 + 12x_1x_2 - 50x_1 + 80x_2$, subject to the constraints $x_1 \geq 0, x_2 \geq 0, x_1 + x_2 \leq 1, 8x_1^2 + x_2^2 \leq 2$, the maximum occurs at $x^0 = [0, 1]^T$. Find $u^0 = [u_1, u_2]^T$ such that (D) holds at x^0, u^0.

Problem 9. (i) State the *Euler–Lagrange* equation satisfied by a minimizer $y(\cdot)$ of the functional

$$I[y(\cdot)] = \int_a^b L(x, y(x), y'(x))\, dx.$$

(ii) Prove that if $L = L(y, z)$ does not depend on the variable x and if $y(\cdot)$ satisfies the Euler–Lagrange equation, then

$$L(y, y') - y'\frac{\partial L}{\partial z}(y, y') \equiv C$$

on the interval $[a, b]$, for some constant C.

Problem 10. Recall that the *Farkas alternative* says that either

(i) $$Ax = b \ \text{ has a solution } x \geq 0$$

or

(ii) $$A^T y \geq 0,\ y \cdot b < 0 \ \text{ has a solution } y,$$

but not both. Use this to prove the following variant: Either

(i') $$Ax \leq b \ \text{ has a solution } x \geq 0$$

or

(ii') $$A^T y \geq 0,\ y \cdot b < 0 \ \text{ has a solution } y \geq 0,$$

but not both.

(Hint: Assertion (i') is equivalent to saying that $Ax + z = b$ has a solution $x \geq 0, z \geq 0$. Rewrite this in the form (i), for the matrix $\tilde{A} = [A\ I]$.)