MATH104 FINAL — L. BARTHOLDI'S CLASS F03

NAME & SID: ____________________

This is the Final Examination for class 104 section 1, to be taken from 9:10 to 12:00 on December 12, 2003. Please hand in only these stapled papersheets. Remember to fill in your name and SID. Good luck!

(1) Let $x = (x_1, x_2, \dots)$ denote a sequence of real numbers. Connect each notion to all definitions equivalent to it:

- The sequence x is convergent if •
- The series $\sum_{n\geq 0} x_n$ is convergent if •
- The series $\sum_{n\geq 0} x_n$ is absolutely convergent if •

- • $\forall \epsilon > 0 : \exists N \in \mathbb{N} : \forall m, n \geq N : |x_n - x_m| < \epsilon$
- • $\forall \epsilon > 0 : \exists N \in \mathbb{N} : \forall k \in \mathbb{N} : |\sum_{i=N}^{N+k} x_i| < \epsilon$
- • $\forall \epsilon > 0 : \exists N \in \mathbb{N} : \forall k \in \mathbb{N} : \exists y \in \mathbb{R} : \sum_{i=N}^{N+k} |x_i| < \epsilon$
- • $\exists y \in \mathbb{R} : \forall \epsilon > 0 : \exists N \in \mathbb{N} : |y - \sum_{i=1}^{N} x_i| < \epsilon$
- • $\exists y \in \mathbb{R} : \forall \epsilon > 0 : \exists N \in \mathbb{N} : \forall n \geq N : |x_n - y| < \epsilon$
- • $\exists y \in \mathbb{R} : \forall \epsilon > 0 : \exists N \in \mathbb{N} : |y - \sum_{i=1}^{N} |x_i|| < \epsilon$
- • $\forall N \in \mathbb{N} : \exists \epsilon > 0 : \forall n \geq N : |x_n| < \epsilon$

(2) Let $f = (f_1, f_2, \dots)$ be a sequence of continuous functions: $[0,1] \to \mathbb{R}$. Check ($\surd$) the definitions that are equivalent to "The sequence f converges uniformly".

- ☐ $\forall x \in [0,1] : \lim_{n\to\infty} f_n(x)$ exists.
- ☐ $\forall x \in [0,1] : \lim_{n\to\infty} f_n(x)$ exists and is continuous.
- ☐ $\forall x \in [0,1] : \exists y \in \mathbb{R} : \forall \epsilon > 0 : \exists N \in \mathbb{N} : \forall n \geq N : |f_n(x) - y| < \epsilon$.
- ☐ $\forall x \in [0,1] : \forall \epsilon > 0 : \exists y \in \mathbb{R} : \exists N \in \mathbb{N} : \forall n \geq N : |f_n(x) - y| < \epsilon$.
- ☐ $\forall x \in [0,1] : \forall \epsilon > 0 : \exists N \in \mathbb{N} : \forall n \geq N : \exists y \in \mathbb{R} : |f_n(x) - y| < \epsilon$.
- ☐ $\forall \epsilon > 0 : \exists N \in \mathbb{N} : \forall x \in [0,1] : \exists y \in \mathbb{R} : \forall n \geq N : |f_n(x) - y| < \epsilon$.
- ☐ $\forall \epsilon > 0 : \exists N \in \mathbb{N} : \forall n \geq N : \forall x \in [0,1] : \exists y \in \mathbb{R} : |f_n(x) - y| < \epsilon$.
- ☐ There exists a function $g : [0,1] \to \mathbb{R}$ such that
$$\forall \epsilon > 0 : \lim_{n\to\infty} \sup_{x\in[0,1]} |g(x) - f_n(x)| = 0.$$

Date: December 12, 2003.

(3) For each of the following, either give an example of such a function, or explain in **at most 2 lines** why it is impossible:

(a) A continuous function $f : [0,1] \to \mathbb{R}$ with $f(0) = f(1)$, that never reaches its maximum.

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(b) A differentiable function $f : [0,1] \to \mathbb{R}$ with $f(0) = f(1)$, but $f'(x) \neq 0$ for all $x \in [0,1]$.

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(c) A continuous function $f : [0,1] \to \mathbb{R}$ which is not the pointwise limit of differentiable functions.

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(d) A continuous function $f : [0,1] \to \mathbb{R}$ which is not the uniform limit of differentiable functions.

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(e) A discontinuous function $f : [0,1] \to \mathbb{R}$ which is the pointwise limit of differentiable functions.

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(f) A discontinuous function $f : [0,1] \to \mathbb{R}$ which is the uniform limit of differentiable functions.

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(4) Consider the function $f : [0,1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} (-1)^n/n & \text{if } x = 1/n \text{ for some } n \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Where is f continuous?

(b) Where is it differentiable?

(c) Is $\lim_{x\to 0} f(x)$ defined? In that case what is its value?

(d) Is f Riemann-integrable? In that case, what is $\int_0^1 f(t)dt$?

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(5) Compute the radius of convergence of the following power series:

$$x + 2x^2 + x^3 + 4x^4 + x^5 + 6x^6 + \cdots$$

$$2x^2 + 2^2x^{2^2} + 2^3x^{2^3} + 2^4x^{2^4} + \cdots$$

$$1 + x + 2x^2 + 3x^3 + 5x^5 + \cdots = \sum_{n=0}^{\infty} f_n x^n,$$

where $f_0 = f_1 = 1$, $f_{n+2} = f_{n+1} + f_n$ are the Fibonacci numbers.

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(6) Recall the theorem asserting that if $f : [a,b] \to \mathbb{R}$ is a continuous function, then it is bounded and reaches its maximum.

(a) Find a function $f : (a,b] \to \mathbb{R}$ that is continuous and reaches its maximum, but is not bounded.

(b) Find a function $f : [a,b] \to \mathbb{R}$ that is discontinuous and reaches its maximum, but is not bounded.

(c) Find a function $f : (a,b] \to \mathbb{R}$ that is continuous and bounded, but does not reach its maximum.

(d) Find a function $f : [a,b] \to \mathbb{R}$ that is discontinuous and bounded, but does not reach its maximum.

(7) Prove that if $f : \mathbb{R} \to \mathbb{R}$ is a continuous function such that $f(f(x)) = 0$ for all $x \in \mathbb{R}$, then there is a non-empty interval $I \subset \mathbb{R}$ such that $f(x) = 0$ for all $x \in I$.